Reduced Dirac Equation And Lamb Shift As An Off-mass-shell Effect In Quantum Electrodynamics

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Based on the precision experimental data of energy-level differences in hydrogenlike atoms, especially the $1S-2S$ transition of hydrogen and deuterium, the necessity of introducing a reduced Dirac equation with reduced mass as the substitution of original electron mass is stressed. Based on new cognition about the essence of special relativity, we provide a reasonable argument for reduced Dirac equation to have two symmetries, the invariance under the (newly defined) space-time inversion and that under the pure space inversion, in a noninertial frame. By using reduced Dirac equation and within the framework of quantum electrodynamics in covariant form, the Lamb shift can be evaluated (at one-loop level) as the radiative correction on a bound electron staying in an off-mass-shell state–a new approach eliminating the infrared divergence. Hence the whole calculation, though with limited accuracy, is simplified, getting rid of all divergences and free of ambiguity.

Keywords: Reduced Dirac Equation, Lamb shift, off-mass-shell

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I. INTRODUCTION

As is well known, the Dirac equation for electron in a hydrogenlike atom is usually treated as a one-body equation with the nucleus being an inert core having infinite mass and exerting a potential $V(r) = -\frac{Z\alpha}{r}$ $(\hbar = c = 1)$ on the electron. Then the rigorous solution of energy levels reads[1]:

$$E_{nj} = m_e f(n,j) \tag{1}$$

$$f(n,j) = \left[1 + \frac{(Z\alpha)^2}{(n-\beta)^2}\right]^{-\frac{1}{2}} \tag{2}$$

$$\beta = j + \frac{1}{2} - \sqrt{(j+\frac{1}{2})^2 - (Z\alpha)^2} \tag{3}$$

where j is the total angular momentum. The expansion of $f(n,j)$ to the power of $(Z\alpha)^6$ is given as [1]

$$\begin{array}{l} f(n,j) = 1 - \frac{(Z\alpha)^2}{2n^2} - \frac{(Z\alpha)^4}{2n^3}\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}\right) \\ -\frac{(Z\alpha)^6}{8n^3}\left[\frac{1}{(j+\frac{1}{2})^3} + \frac{3}{n(j+\frac{1}{2})^2} + \frac{5}{2n^3} - \frac{6}{n^2(j+\frac{1}{2})}\right] + \cdots \end{array} \tag{4}$$

Obviously, besides the rest energy of the electron given by the first term, the second term has exactly the form of Bohr energy level except that the mass m_e must be replaced by the reduced mass

$$\mu = \frac{m_e m_N}{m_e + m_N} \equiv \frac{m_e m_N}{M} \tag{5}$$

with m_N being the mass of the nucleus and $M = m_e + m_N$.

However, as discussed in Refs.[1] and [2], the concept of reduced mass in relativistic quantum mechanics (RQM) is ambiguous to some extent. Beginning from 1950's, a number of authors have been devoting a great effort at the level of two-body RQM and that of

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quantum electrodynamics (QED) to take account of the recoil effect [1, 3, 4], incorporating their results in a compact form (to order of α^4):

$$E = M + \mu[f(n,j) - 1] - \frac{\mu^2}{2M}[f(n,j) - 1]^2 + \frac{(Z\alpha)^4\mu^3}{2n^3m_N^2}\left[\frac{1}{j+\frac{1}{2}} - \frac{1}{l+\frac{1}{2}}\right](1 - \delta_{l0}) \qquad (6)$$

A comprehensive review on the theory of hydrogenlike atoms can be found in Ref.[27]. The aim of this paper is two-fold: First, based on the experimental data of hydrogen $1S - 2S$ transition frequency [5] and its isotope shift of hydrogen and deuterium [6], we stress the necessity of the introduction of reduced mass μ (section II) before we are able to argue the reasonableness of introducing a "reduced Dirac equation" with μ as the substitution of m_e (section III). Second, based on above conception, we will present a calculation of Lamb Shift (LS) as an off-mass-shell effect by performing the evaluation of self-energy diagrams of electron (section IV) and photon (section V) as well as the vertex function (section VI) at the one-loop level of QED in covariant form. The new insight of our calculation is focused on the regularization renormalization method (RRM). As initiated by J-F Yang [7] and elaborated in a series of papers ([9, 24–26, 8a, 8b] and references therein), we can get rid of all ultra violet divergences in the calculation of quantum field theory (QFT). Furthermore, in this paper, we will be able to get rid of the annoying infrared divergence in the vertex function by treating the electron moving off its mass-shell to certain extent which is fixed through the evaluation of self-energy diagram or by the Virial theorem. Based on above improvements, the one-loop calculation yields values of LS in a simple but semi-quantitative way (section VII and VIII). Although the accuracy is limited at one-loop level, we hope our approach could be served as a new starting point for calculations at high-loop order to get accurate results at a comparably low labor cost. The final section IX and Appendix will contain a summary and discussion.

II. THE $1S - 2S$ TRANSITION OF ATOMIC HYDROGEN AND DEUTERIUM

In the last decade, thanks to remarkable advances in high resolution laser spectroscopy and optical frequency metrology, the $1S - 2S$ two-photon transition in atomic hydrogen H (or deuterium D) with its natural linewidth of only $1.3Hz$ had been measured to a very high precision. In 1997, Udem *et al.*determined the $1S - 2S$ interval of H being [5]

$$f^{(H)}(1S-2S) = 2466061413187.34(84) \quad kHz \tag{7}$$

Even four years earlier, Schmidt-Kalar *et al.* measured the isotope-shift of the $1S-2S$ transition of H and D to an accuracy of 3.7×10^{-8}[6], giving (as quoted in [10]):

$$\Delta f \equiv f^{(D)}(2S-1S) - f^{(H)}(2S-1S) = 670994337(22) \quad kHz \tag{8}$$

(In 1998, Huber *et al.*measured a more accurate data [28]: 670994334.64(15) kHz). which is of the order of 10^{-4} in comparison with Eq.(7). As pointed out in Ref.[6], this 671 GHz isotope-shift can be ascribed almost entirely to the different masses of proton (p) and deuteron (d). And the nuclear volume effects become important because the QED effects cancel considerably in the isotope shift.

Here, we wish to emphasize that in the first approximation, both experimental data (7) and (8) can be well accounted for by simply resorting to Eq.(1) with m_e replaced by the reduced mass

$$\mu_H = \frac{m_e m_p}{m_e+m_p}, \quad \mu_D = \frac{m_e m_d}{m_e+m_d} \tag{9}$$

for H and D respectively.

Indeed, adopting the following updated values [10–13]

$$\alpha = (137.03599944)^{-1}, \quad \alpha^2 = 0.532513542\times10^{-4} \tag{10}$$

$$\alpha^4 = 0.283570673\times10^{-8}, \quad \alpha^6 = 0.151005223\times10^{-12} \tag{11}$$

$$m_e = 0.51099906 \quad MeV = 1.2355897\times10^{20} \quad Hz \tag{12}$$

$$R_\infty = \frac{1}{2}\alpha^2 m_e = 3.28984124\times10^{15} \quad Hz \tag{13}$$

$$\frac{m_p}{m_e} = 1836.1526665 \tag{14}$$

and denoting

$$\frac{m_e}{m_p} = b_H = 5.446170255\times10^{-4}, \quad \frac{1}{1+b_H} = 0.999455679 \tag{15}$$

$$\frac{m_e}{m_d} = b_D = 2.724436319\times10^{-4}, \quad \frac{1}{1+b_D} = 0.99972763 \tag{16}$$

we can calculate the energy difference of $2S$ and $1S$ of H through Eq.(1) with m_e replaced by μ_H (the superscript RDE refers to the reduced Dirac equation)

$$\begin{aligned}\Delta E_H^{RDE}(2S-1S) &= \mu_H[f(2,1/2)-f(1,1/2)] \\ &= \frac{m_e}{1+b_H}(1.996950159\times 10^{-5}) \\ &= 1.2355897\times 10^{20}\times 0.999455679\times 1.996950159\times 10^{-5} \\ &= 2.466067984\times 10^{15}\ \ Hz \end{aligned} \tag{17}$$

which is only a bit larger than the experimental data Eq.(7) with accuracy 3×10^{-6}. However, a more stringent test of RDE should be the isotope shift of H and D. We have

$$\frac{1}{1+b_D}-\frac{1}{1+b_H}=(b_H-b_D)-(b_H^2-b_D^2)+(b_H^3-b_D^3)+\cdots=2.719511528\times 10^{-4} \tag{18}$$

$$\Delta E_{D-H}^{RDE}=(\mu_D-\mu_H)[f(2,1/2)-f(1,1/2)]=6.7101527879\times 10^{11}\ \ Hz \tag{19}$$

which has only a discrepancy larger than the experimental data, Eq.(8) by 20.941 MHz with accuracy 3×10^{-5}. Of course, it is still not satisfied in an analysis of high precision [6]. Let us resort to the Eq.(6), where the third term does provide a further modification:

$$\begin{aligned} &-\ \frac{1}{2}m_e[\frac{b_D}{(1+b_D)^3}-\frac{b_H}{(1+b_H)^3}]\{[f(2,1/2)-1]^2-[f(1,1/2)-1]^2\} \\ &=\ \frac{1}{2}m_e[(b_H-b_D)-3(b_H^2-b_D^2)+\cdots](-6.646361554\times 10^{-10})=-11.176\ MHz \end{aligned} \tag{20}$$

which brings the discrepancy between the theory and experimental down to less than 10 MHz.

Although the detail explanation for this discrepancy remains quite complicated[6], the above comparison is enough to convince us that the inevitable appearance of reduced mass in the RDE or Eq.(6) is by no means a simple fortune. It must have a deep reason from a theoretical point of view. Notice further that once the conditions $m_e \ll m_p$ and $m_e \ll m_d$ hold, the difference of spin between p and d seems not so important. So in next section, we will strive to justify the reduced Dirac equation on a reasonable basis. Of course, it is still an approximate one, but seems much better than the original Dirac equation when dealing with hydrogenlike atoms.

III. REDUCED MASS AND REDUCED DIRAC EQUATION

Consider a system of two particles with rest masses m_1 and m_2. Their coordinates in the center-of-mass (CM) system are $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively, as shown in Fig.1. If there is a potential $V(r) = V(|\mathbf{r}_1 - r_2|)$ between them, two equations $m_1\ddot{\mathbf{r}}_1 = -\nabla_r V(r)$ and $m_2\ddot{\mathbf{r}}_2 = \nabla_r V(r)$ will reduce to one:

$$\mu\frac{d^2\mathbf{r}}{dt^2} = -\nabla_r V(r), \quad (\mu = \frac{m_1 m_2}{m_1 + m_2}) \tag{21}$$

At first sight, the definition of center-of -mass (CM) in classical mechanics $m_1 r_1 = m_2 r_2$ becomes doubtful in the theory of special relativity (SR) because the mass is no longer a constant. But actually, we can still introduce the coordinate of CM in the laboratory coordinate system (LCS) (with $\mathbf{r}_1'$ and $\mathbf{r}_2'$ being the coordinates of m_1 and m_2):

$$R = \frac{1}{M}(m_1\mathbf{r}_1' + m_2\mathbf{r}_2') = (X, Y, Z), \quad (M = m_1 + m_2) \tag{22}$$

and the relative coordinate of m_1 and m_2 ($\mathbf{r}_i = \mathbf{r}_i' - \mathbf{R}$, $i = 1, 2$) :

$$\mathbf{r} = \mathbf{r}_1' - \mathbf{r}_2' = \mathbf{r}_1 - \mathbf{r}_2 = (x, y, z) \tag{23}$$

Here the motion of CM in the LCS is assumed to be slow and so

$$\frac{\partial}{\partial x_1'} = \frac{m_1}{M}\frac{\partial}{\partial X} + \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial x_2'} = \frac{m_2}{M}\frac{\partial}{\partial X} - \frac{\partial}{\partial x} \tag{24}$$

Notice that the momentum $\mathbf{P}$ of CM and the relative momentum $\mathbf{p}_r$ becomes operator in quantum mechanics (QM) without explicit dependence on mass:

$$\mathbf{P} = -i\hbar\nabla_{\mathbf{R}}, \quad \mathbf{p}_r = -i\hbar\nabla_{\mathbf{r}} \tag{25}$$

Thus the momenta of m_1 and m_2 in laboratory coordinate system (LCS) read:

$$\mathbf{p}_1' = -i\hbar\nabla_{\mathbf{r}_1'} = \frac{m_1}{M}\mathbf{P} + \mathbf{p}_r, \quad \mathbf{p}_2' = -i\hbar\nabla_{\mathbf{r}_2'} = \frac{m_2}{M}\mathbf{P} - \mathbf{p}_r \tag{26}$$

Since the center-of-mass coordinate system (CMCS) is also an inertial frame which can be transformed from the LCS via a linear Lorentz transformation, it is defined by the condition that $\mathbf{P} = 0$ in CMCS. In other words, CMCS is defined by the condition $\mathbf{p}_1 + \mathbf{p}_2 = 0$, or from Eq.(26):

$$\mathbf{p}_1' = -i\hbar\nabla_{\mathbf{r}_1'} = \mathbf{p}_r, \quad \mathbf{p}_2' = -i\hbar\nabla_{\mathbf{r}_2'} = -\mathbf{p}_r \tag{27}$$

Evidently, the above definition of CMCS remains valid in the realm of relativistic QM (RQM) even the exact meaning of CM seems obscure to some extent due to the conjugation relation of a particle's position and its momentum, see Fig.1.

Now, from Eq.(27), it is natural to replace $\mathbf{p}_1$ and $\mathbf{p}_2$ by $\mathbf{p}_r$, reducing the two-particle degrees of freedom to one. In the meantime, the origin of CMCS is discarded, it is substituted by the position of m_2 ($\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$). We will call the system associated with $\mathbf{r}$ the relative motion coordinate system (RMCS), which should be viewed as a deformation of CMCS. The transformation from CMCS to RMCS is by no means a linear one. Rather, the origin of RMCS (m_2) is moving non-uniformly in the CMCS. Therefore, while rest masses m_1 and m_2 remain the same in both LCS and CMCS, they reduce to one mass $\mu = \frac{m_1 m_2}{m_1+m_2}$ for m_1 in RMCS (or for m_2 if m_1 is chosen as the origin of RMCS).

Let us express the total energy $E = E_1 + E_2$ in CMCS in terms of p_r and reduced mass μ ($\mu = \frac{m_1 m_2}{M}$, $M = m_1 + m_2$), where

$$E_1 = \sqrt{m_1^2 + p_1^2} = \sqrt{m_1^2 + p_r^2}, \quad E_2 = \sqrt{m_2^2 + p_2^2} = \sqrt{m_2^2 + p_r^2} \tag{28}$$

Treating all p_1, p_2 and p_r being c-numbers, we have

$$E^2 = (E_1 + E_2)^2 = M^2 + \frac{M}{\mu}p_r^2 + \frac{1}{4\mu^2}p_r^4(4 - \frac{M}{\mu}) + \cdots \tag{29}$$

where the expansion in p_r is kept to the order of p_r^4. Two extreme cases will be considered separately:

A. $m_2 \gg m_1$, $\mu \lesssim m_1$, $M \gg \mu$:

$$\begin{aligned} E^2 &= M^2\left[1 + \frac{1}{\mu M}p_r^2 - \frac{1}{4M\mu^3}p_r^4(1 - \frac{4\mu}{M}) + \cdots\right] \\ E &= M\left[1 + \frac{1}{2\mu M}p_r^2 - \frac{1}{8M\mu^3}p_r^4(1 - \frac{3\mu}{M}) + \cdots\right] \\ &= M + \frac{1}{2\mu}p_r^2 - \frac{1}{8\mu^3}p_r^4 + \cdots \\ &\simeq M - \mu + \sqrt{\mu^2 + p_r^2} \simeq m_2 + (m_1 - \mu) + \sqrt{\mu^2 + p_r^2} \end{aligned} \tag{30}$$

$$E' \equiv E - m_2 = (m_1 - \mu) + \sqrt{\mu^2 + p_r^2} \tag{31}$$

B. $m_1 = m_2 = m, \quad \mu = \frac{m}{2}, \quad M = 2m = 4\mu$ Then to the accuracy of p_r^4, we have :

$$\begin{aligned} E^2 &= M^2 + \frac{M}{\mu}p_r^2 = 4m^2 + 4p_r^2 \\ E &= 2m + \frac{1}{2\mu}p_r^2 - \frac{1}{32\mu^3}p_r^4 + \cdots \\ E' &\equiv E - M = \frac{1}{2\mu}p_r^2 - \frac{1}{32\mu^3}p_r^4 \simeq \frac{1}{2\mu}p_r^2, \quad (\text{if } \ p_r^2 \ll \mu^2) \end{aligned} \tag{32}$$

It is interesting to see that after introducing μ and p_r, the energy E' in RMCS looks quite "relativistic" in the case A whereas it looks rather "non-relativistic" in the case B even both of them are derived from the relativistic expressions, Eq.(28), approximately.

Since the RMCS is not an inertial system, the original mass of m_1 in CM changes abruptly to μ as shown in Eq.(31). How can we derive the reduced Dirac equation (RDE) in RMCS? Fortunately, we already found a basic symmetry, the space-time inversion symmetry, which not only serves as the essence of special relativity (SR), but also goes beyond it to derive the original Dirac equation and the tachyon theory for neutrinos [14–17]. Based on this symmetry, we are going to derive the equation in RQM for case either A or B respectively.

Let us consider case B ($m_1 \simeq m_2$) first. The motivation is stemming from the success of using the Schrödinger equation to heavy-quarkoniums like $c\bar{c}$ and $b\bar{b}$ in particle physics ([18], see also [15] §9.5 D). Ignoring the spin of both m_1 and m_2, we assume the coupling equations in laboratory system for the two-particle system as:

$$\begin{cases} i\hbar\frac{\partial\varphi}{\partial t} &= (m_1+m_2)c^2\varphi + V(|\mathbf{r}'_1 - \mathbf{r}'_2|)(\varphi+\chi) - (\frac{\hbar^2}{2m_1}\nabla^2_{\mathbf{r}'_1} + \frac{\hbar^2}{2m_2}\nabla^2_{\mathbf{r}'_2})(\varphi+\chi) \\ i\hbar\frac{\partial\chi}{\partial t} &= -(m_1+m_2)c^2\chi - V(|\mathbf{r}'_1 - \mathbf{r}'_2|)(\varphi+\chi) + (\frac{\hbar^2}{2m_1}\nabla^2_{\mathbf{r}'_1} + \frac{\hbar^2}{2m_2}\nabla^2_{\mathbf{r}'_2})(\varphi+\chi) \end{cases} \tag{33}$$

where $\varphi = \varphi(\mathbf{r}'_1, \mathbf{r}'_2, t)$ and $\chi = \chi(\mathbf{r}'_1, \mathbf{r}'_2, t)$ are hidden "particle" and "antiparticle" fields of the two-particle system (From now on, the $\mathbf{r}'_i (i = 1, 2)$ is the flowing coordinate of "fields" in QM, *i.e.*, that of "fictitious point particles". See Fig.1). Eq.(33) remains invariant under the (newly defined) space-time inversion ($\mathbf{r}'_1 \to -\mathbf{r}'_1, \mathbf{r}'_2 \to -\mathbf{r}'_2, t \to -t$):

$$\begin{cases} \varphi(-\mathbf{r}'_1, -\mathbf{r}'_2, -t) &\longrightarrow \chi(\mathbf{r}'_1, \mathbf{r}'_2, t) \\ \chi(-\mathbf{r}'_1, -\mathbf{r}'_2, -t) &\longrightarrow \varphi(\mathbf{r}'_1, \mathbf{r}'_2, t) \end{cases} \tag{34}$$

$$V(-\mathbf{r}'_1, -\mathbf{r}'_2, -t) \longrightarrow V(\mathbf{r}'_1, \mathbf{r}'_2, t) \tag{35}$$

Note that, however, the time t is not contained in V explicitly. Eq.(35) merely means that both m_1 and m_2 ($m_1 \approx m_2$) transform into their antiparticles under the space-time inversion. Actually, the hidden antiparticle field χ enhances in nearly equal strength in m_1 and m_2 when fictitious particles' velocities increase with the enhancement of attractive potential $V(r)$.

After introducing the CM coordinate $\mathbf{R} = \frac{1}{M}(m_1\mathbf{r'}_1 + m_2\mathbf{r'}_2)$, $(M = m_1 + m_2)$ and relative coordinate $\mathbf{r} = \mathbf{r}'_1 - \mathbf{r}'_2$, and setting

$$\varphi = \Phi + i\frac{\hbar}{Mc^2}\dot{\Phi}, \qquad \chi = \Phi - i\frac{\hbar}{Mc^2}\dot{\Phi} \tag{36}$$

we find ($\mu = \frac{m_1 m_2}{M}$)

$$\ddot{\Phi} - c^2\nabla_R^2\Phi - c^2\frac{M}{\mu}\nabla_r^2\Phi + \frac{1}{\hbar^2}(M^2c^4 + 2VMc^2)\Phi = 0 \tag{37}$$

Its stationary solution reads

$$\Phi(\mathbf{R}, \mathbf{r}, t) = \psi(\mathbf{r}) \exp\left[\frac{i}{\hbar}(\mathbf{P}\cdot\mathbf{R} - Et)\right] \tag{38}$$

where E is the total energy of the system while $\mathbf{P}$ the momentum of CM. The reduced "one-body" equation for $\psi(\mathbf{r})$ turns out to be: [$*$]

$$\begin{cases} \left[-\dfrac{\hbar^2}{2\mu}\nabla_{\mathbf{r}}^2 + V(\mathbf{r})\right]\psi(\mathbf{r}) = \varepsilon\psi(\mathbf{r}) \\ \varepsilon = \dfrac{1}{2Mc^2}(E^2 - M^2c^4 - \mathbf{P}^2c^2) \end{cases} \tag{39}$$

We set $\mathbf{P} = 0$ (*i.e.*turn to CMCS) and denote the binding energy $B = Mc^2 - E$, yielding:

$$B = Mc^2\left[1 - (1 + \frac{2\varepsilon}{Mc^2})^{1/2}\right] = -\varepsilon + \frac{1}{2}\frac{\varepsilon^2}{Mc^2} - \cdots \tag{40}$$

[$*$] With Eq.(35), Eq.(37) is invariant under the space-time inversion ($\mathbf{r} \to -\mathbf{r}, t \to -t$). Equivalently, under the mass inversion ($m_1 \to -m_1, m_2 \to -m_2$), Eq.(37) and Eq.(39) remain invariant in the sense that not only $\mu \to -\mu, M \to -M$, but also $V(\mathbf{r}) \to -V(\mathbf{r}), \varepsilon \to -\varepsilon$. Notice that, however, the simultaneous inversion of m_1 and m_2 implies $m_1 \simeq m_2$, so both particles change under their mutual interaction $V(\mathbf{r})$ simultaneously. Here V, being the "internal potential energy" of two-body system, was called as a "scalar potential". We see that either the invariance under the space-time inversion or that under the mass inversion is capable of showing the particle-antiparticle symmetry (*i.e.*, relativistic nature) of a system essentially.

Notice that although Eq.(39) looks like a "non-relativistic" stationary Schrödinger equation, it is essentially relativistic. This can be seen from its remarkable property that the eigenvalue ε has a lower bound $-\frac{1}{2}Mc^2$, corresponding to $E_{\min}=0$ ($B_{\max}=M$)!

An example is: consider "positronium" composed of e^+ and e^- with charge Ze and $-Ze$ respectively. Once when the "fictitious charge number" Z increases from 1 to $Z_{max}=(\frac{4}{\alpha^2})^{1/4}=16.555$, the whole bound system would have lowest ground energy $E_{min}=0$! So Eq.(39) is really a relativistic QM equation capable of giving a nonperturbative solution under the strong coupling.

Eq.(39) provides a justification (realization) of conjecture Eq.(32) relevant to case B ($m_1\simeq m_2$) where the spin of both particles is merely of second importance.

Now let us turn to case A where $m_2\gg m_1$, taking the spin of m_1 into account but ignoring that of m_2 as before. Based on the experience in case B, also because of great difficulty to derive the equation starting from the laboratory system for this case A, we directly introduce the reduced Dirac equation (RDE) in the RMCS as a pair of coupled equations of two-component spinors $\varphi(\mathbf{r},t)$ and $\chi(\mathbf{r},t)$, ($c=\hbar=1$)

$$\begin{cases} i\dot{\varphi} &= i\sigma_1\cdot\nabla_{\mathbf{r}}\chi+\mu\varphi+V(\mathbf{r})\varphi \\ i\dot{\chi} &= i\sigma_1\cdot\nabla_{\mathbf{r}}\varphi-\mu\chi+V(\mathbf{r})\chi \end{cases} \tag{41}$$

with μ replacing m_1. Here σ_1 are Pauli matrices acting on the spin space of particle m_1. Eq.(41) is invariant under the space-time inversion ($\mathbf{r}\to-\mathbf{r},t\to-t$), $\varphi(-\mathbf{r},-t)\to\chi(\mathbf{r},t)$, $\chi(-\mathbf{r},-t)\to\varphi(\mathbf{r},t)$ whereas we assume

$$V(-\mathbf{r},-t)\longrightarrow -V(\mathbf{r},t) \tag{42}$$

here in contrast to Eq.(35) for the case B. [†]

The reasons are as follows: (a) Eq.(41) should degenerate into the original Dirac equation when $m_2\to\infty$, $\mu=\frac{m_1m_2}{m_1+m_2}\to m_1$. (b) Since now $m_2\gg m_1$ (but $m_2\neq\infty$), m_1 is moving

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[†] For a hydrogenlike atom, $V(r)=-\frac{Ze^2}{r}$ does not contain time t explicitly. Eq.(42) merely means that under the space-time inversion, the electron transforms into a position whereas the nucleus remains unchanged. See point (a) of section IX. Previously, the V in Eq.(42) was called as a "vector potential", meaning the "potential energy" of the electron in an "external field" of nucleus. Note that, formally, Eq.(41) remains invariant under a mass inversion as $\mu\to-\mu,\phi\to\chi,\chi\to\phi$ ($V(\mathbf{r})$ remains unchanged) in a noninertial frame $RMCS$. Actually, since $m_1=m_e\to-m_e$, but $m_2=m_N\to m_N,\mu\to-\mu(1+\frac{2m_e}{M})$. So Eq.(41) has an inaccuracy up to $\frac{2m_e}{M}$ ($<1.1\times10^{-3}$ for H).

much faster than m_2 in the CMCS. Hence the antiparticle field χ enhances much appreciably in m_1 than that in m_2, a situation totally different from that in the case B where $m_1 \approx m_2$. (c) If instead of Eq.(42), we still assume $V(-\mathbf{r}, -t) \longrightarrow V(\mathbf{r}, t)$ like Eq.(35) and change the sign before $V(\mathbf{r})$ in the second equation of Eq.(41) to keep its invariance under the space-time inversion, then we would get an equation which would lead to a reversed fine-structure of atom (*e.g.*, the $P_{1/2}$ state would lie above the $P_{3/2}$ state), a wrong prediction obviously excluded by experiments.

However, one kind of invariance is not enough to fix an equation. Indeed, the beauty of Dirac equation or RDE is hidden in two symmetries: besides the symmetry of space-time inversion, it has another left-right (parity) symmetry. To see it, we define

$$\xi = \frac{1}{\sqrt{2}}(\varphi + \chi), \quad \eta = \frac{1}{\sqrt{2}}(\varphi - \chi) \tag{43}$$

and recast Eq.(41) into:

$$\begin{cases} i\dot{\xi} &= i\sigma_1 \cdot \nabla_{\mathbf{r}}\xi + \mu\eta + V(\mathbf{r})\xi \\ i\dot{\eta} &= -i\sigma_1 \cdot \nabla_{\mathbf{r}}\eta + \mu\xi + V(\mathbf{r})\eta \end{cases} \tag{44}$$

which is invariant under a pure space inversion ($\mathbf{r} \to -\mathbf{r}, t \to t$) if assuming

$$\xi(-\mathbf{r}, t) \to \eta(\mathbf{r}, t), \quad \eta(-\mathbf{r}, t) \to \xi(\mathbf{r}, t), \quad V(-\mathbf{r}) \to V(\mathbf{r}) = V(r) \tag{45}$$

The parity invariance of Dirac equation or RDE has a far-reaching consequence that the Dirac particle is always a subluminal one. By contrast, once the parity is violated to maximum, a superluminal particle (tachyon) will emerge. Interestingly enough, any theory capable of treating particle and antiparticle on an equal footing must respect to the common basic symmetry—the invariance of space-time inversion. The new insight of this section is this symmetry can be applied even in a noninertial frame—the RMCS. Of course, the validity of RDE can only be verified by experiments as discussed in section II, although it is still an approximate description of nature like any other theory in physics. For further discussion, see section IX.

IV. SELF-ENERGY CORRECTION OF A BOUND ELECTRON IN ATOM

In our understanding, one important reason why the calculations of QED for electron in a hydrogenlike atom is so complicated lies in the fact that while calculations are performed in the CMCS, the center of potential (the nucleus with mass $m_2 = m_N$) undergoes a complex motion. So the recoil effect interwinds with the high-loop correction of QED, as discussed in many chapters of the books [1] and [2]. We will try to find an alternative approach by adopting the RDE and doing calculation in the RMCS. Let us begin with the Feynman diagram integral (FDI) of electron self-energy at one-loop level, adopting the Bjorken-Drell metric and rationalized Gaussian units with electron charge $-e(e > 0)$, see Fig.2(a) ([8a]).

$$-i\Sigma(p) = (ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{g_{\mu\nu}}{ik^2} \gamma^\mu \frac{i}{\not{p} - \not{k} - \mu} \gamma^\nu \tag{46}$$

Here a free electron with reduced mass μ is moving at a four-dimensional momentum p, whose spatial component is just the relative momentum $\mathbf{p}_r$ discussed in the previous section, k is the momentum of virtual photon. As usual, a Feynman parameter x will bring Eq.(46) into

$$-i\Sigma(p) = -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{N}{D} \tag{47}$$

$$\frac{1}{D} = \int_0^1 \frac{dx}{[k^2 - 2p \cdot kx + (p^2 - \mu^2)x]^2}, \quad N = -2(\not{p} - \not{k}) + 4\mu \tag{48}$$

($\not{p} = p^\mu \gamma_\mu,\ p \cdot k = p^\mu k_\mu$). A shift in momentum integration:$k \to K = k - xp$ recast Eq.(47) into

$$-i\Sigma(p) = -e^2 \int_0^1 dx[-2(1-x)\not{p} + 4\mu]I \tag{49}$$

with a logarithmically divergent integral (in Minkowski momentum space):

$$I = \int \frac{d^4K}{(2\pi)^4} \frac{1}{[K^2 - M^2]^2}, \quad M^2 = p^2x^2 + (\mu^2 - p^2)x \tag{50}$$

Our new regularization-renormalization method (RRM) is based on a cognition that the virtual process in the self-energy diagram does provide a radiative correction to the electron mass but only when the electron is off the mass shell, *i.e.*, $p^2 \neq \mu^2$. When it is on the mass

shell, $p^2 = \mu^2$, the appearance of a divergent integral like I in Eq.(50) is essentially a warning on the fact that to calculate the mass of electron is beyond the ability of perturbative QED.

Let us consider the converse: if $\Sigma(p)$ does modify the electron mass μ to some extent, it must comes from the divergent integral I. However, the latter is a dimensionless number, we can change the unit of M (and k) at our disposal without any change in the value of I. So any real change of μ (on the mass shell) is incredible. The deeper reason lies in a "principle of relativity" in epistemology: everything is moving and becomes recognizable only in relationship with other things. What we can understand is either no mass scale or two mass scales, but never one mass scale. For instance, in the famous Gross-Neveu model [19], a massive fermion is created only in accompanying with the change (phase transition) of its environment (vacuum) which provides another mass scale (a standard weight). Another example is just the change of electron mass from m_e to μ_H in a hydrogen atom due to the coexistence of atom nucleus—the proton, this change is also a nonperturbative effect.

Therefore, we expected too much in the past. There is no way to evaluate Eq.(50) unambiguously or pick out some finite and fixed modification on the mass μ. What we can do is to separate the valuable information carried by Eq.(50) from an arbitrary constant which will be introduced by a simple trick and then fixed by the experimental data of μ. We will see the information telling us exactly how the value of I changes when the electron is moving off the mass shell.

To handle Eq.(50), we perform a differentiation with respect to the mass-square parameter M^2, then the integration with respect to K becomes convergent, yielding:

$$\frac{\partial I}{\partial M^2} = \frac{-i}{(4\pi)^2}\frac{1}{M^2} \tag{51}$$

which tells us that while the exact value of I remains obscure, its change linked with M^2 has a definite meaning. So we reintegrate Eq.(51) with respect to M^2 and arrive at

$$I = \frac{-i}{(4\pi)^2}(\ln M^2 + C_1) = \frac{-i}{(4\pi)^2}\ln\frac{M^2}{\mu_2^2} \tag{52}$$

where an arbitrary constant $C_1 = -\ln\mu_2^2$ is introduced (μ_2 should not be confused with the reduced mass μ). Further integration with respect to Feynman parameter x leads to

$$\begin{aligned}\Sigma(p) &= A + B\,\not{p} \\ A &= \tfrac{\alpha}{\pi}\mu\left[2 - 2\ln\tfrac{\mu}{\mu_2} + \tfrac{(\mu^2-p^2)}{p^2}\ln\tfrac{(\mu^2-p^2)}{\mu^2}\right] \\ B &= \tfrac{\alpha}{4\pi}\left[2\ln\tfrac{\mu}{\mu_2} - 3 - \tfrac{(\mu^2-p^2)}{p^2}\left[1 + \tfrac{(\mu^2+p^2)}{p^2}\ln\tfrac{(\mu^2-p^2)}{\mu^2}\right]\right]\end{aligned} \tag{53}$$

Using the chain approximation, we can derive the modification of electron propagator as

$$\frac{i}{\not{p} - \mu} \to \frac{i}{\not{p} - \mu}\frac{1}{1 - \frac{\Sigma(p)}{p\!\!\!/ - \mu}} = \frac{iZ_2}{\not{p} - \mu_R} \tag{54}$$

where

$$Z_2 = \frac{1}{1 - B} \tag{55}$$

is the renormalization factor for wave function of electron and

$$\mu_R = \frac{\mu + A}{1 - B} \tag{56}$$

is the renormalized mass of μ. The increment of mass reads

$$\delta\mu = \mu_R - \mu = \frac{A + \mu B}{1 - B} \tag{57}$$

For a free electron (in the atom), the mass-shell condition $p^2 = \mu^2$ should lead to

$$\delta\mu|_{p^2=\mu^2} = \frac{\alpha\mu}{4\pi}(5 - 6\ln\frac{\mu}{\mu_2}) = 0 \tag{58}$$

as discussed above[‡]. So we must set $\mu_2 = \mu e^{-5/6}$ which in turn fixes

$$Z_2|_{p^2=\mu^2} = \frac{1}{1 + \frac{\alpha}{3\pi}} \approx 1 - \frac{\alpha}{3\pi} \tag{59}$$

However, the above evaluation further provides us with important knowledge of $\delta\mu$ when electron is moving off the mass-shell. Consider the similar diagram in Fig.(2b), we can set on an average meaning that

$$p^2 = \mu^2(1 - \zeta) \tag{60}$$

[‡] We will keep the same mass symbol μ through out high-loop calculations of QED and reconfirm (renormalize) it at every step by experiment. Just like one has to reconfirm his plane ticket before his departure from the airport, he must use the same name through out his entire journey [8b].

with $\zeta > 0$, which implies from Eq.(57) with Eq.(53) that [20]:

$$\delta\mu = \frac{\alpha\mu}{4\pi}\frac{(-\zeta + 2\zeta\ln\zeta)}{1+\alpha/3\pi} \tag{61}$$

where some terms of the order of ζ^2 or $\zeta^2\ln\zeta$ are neglected since $\zeta \ll 1$. Eq.(61) establishes the correspondence between the mass modification $\delta\mu$ and the parameter ζ describing the off-mass-shell extent of electron in the bound state. For a hydrogenlike atom, we may ascribe $\delta\mu$ to the (minus) binding energy of electron in the Bohr theory:

$$\delta\mu = \varepsilon_n = -\frac{Z^2\alpha^2}{2n^2}\mu \tag{62}$$

Then Eq.(61) gives the value of ζ for fixed values Z and n. We will see from the vertex function that these values of ζ are crucial to the calculation of Lamb shift (sections VII and VIII).

V. PHOTON SELF-ENERGY

As discussed in various text books [21–23], we encounter the FDI of vacuum polarization Fig.2(c) as [8a]:

$$\Pi_{\mu\nu}(q) = -(-ie)^2 Tr\int\frac{d^4\bar{p}}{(2\pi)^4}\gamma_\mu\frac{i}{\bar{p}\!\!\!/ - m}\gamma_\nu\frac{i}{\bar{k}\!\!\!/ + q\!\!\!/ - m} \tag{63}$$

Here q is the momentum transfer along the photon line and m the mass of electron. Introducing the Feynman parameter x as in previous section and performing a shift in momentum integration: $\bar{p} \to K = \bar{p} + xq$, we get:

$$\Pi_{\mu\nu}(q) = -4e^2\int_0^1 dx(I_1 + I_2) \tag{64}$$

where

$$I_1 = \int\frac{d^4K}{(2\pi)^4}\frac{2K_\mu K_\nu - g_{\mu\nu}K^2}{(K^2 - M^2)^2} \tag{65}$$

with

$$M^2 = m^2 + q^2(x^2 - x) \tag{66}$$

is quadratically divergent while

$$I_2 = \int \frac{d^4K}{(2\pi)^4} \frac{(x^2-x)(2q_\mu q_\nu - g_{\mu\nu}q^2) + m^2 g_{\mu\nu}}{(K^2-M^2)^2} \tag{67}$$

is only logarithmically divergent like that in Eq.(50). An elegant way to handle I_1, Eq.(65), is modifying M^2 to

$$M^2(\sigma) = m^2 + q^2(x^2-x) + \sigma \tag{68}$$

and differentiating I_1 with respect to σ. After integration with respect to K, we reintegrate it with respect to σ twice, arriving at the limit $\sigma \to 0$:

$$I_1 = \frac{i g_{\mu\nu}}{(4\pi)^2} \left\{ [m^2 + q^2(x^2-x)] \ln \frac{m^2 + q^2(x^2-x)}{\mu_3^2} + C_2 \right\} \tag{69}$$

with two arbitrary constant: $C_1 = -\ln \mu_3^2$ and C_2. Combining I_1 and I_2 together, we find:

$$\Pi_{\mu\nu}(q) = \frac{8ie^2}{(4\pi)^2}(q_\mu q_\nu - g_{\mu\nu}q^2) \int_0^1 dx (x^2-x) \ln \frac{m^2 + q^2(x^2-x)}{\mu_3^2} - \frac{4ie^2}{(4\pi)^2} g_{\mu\nu} C_2 \tag{70}$$

The continuity equation of current induced in the vacuum polarization [21]

$$q^\mu \Pi_{\mu\nu}(q) = 0 \tag{71}$$

is ensured by the factor $(q_\mu q_\nu - g_{\mu\nu}q^2)$. So we set $C_2 = 0$. Consider the scattering between two electrons via the exchange of a photon with momentum transfer $q \to 0$. Adding the contribution of $\Pi_{\mu\nu}(q)$ to the tree diagram amounts to modify the charge square:

$$e^2 \longrightarrow e_R^2 = Z_3 e^2, \quad Z_3 = 1 + \frac{\alpha}{3\pi}(\ln \frac{m^2}{\mu_3^2} - \frac{q^2}{5m^2} + \cdots) \tag{72}$$

As in Ref.[8b], we will set $\mu_3 = m$ so that at the Thomson limit:$\lim_{q\to 0} e_R^2 = e^2$. However, for the purpose of calculating Lamb shift (LS) below, the second term in the parenthesis of Z_3 is important because for a bound state it contributes a term of effective potential (adding to Coulomb potential), called the Uehling potential ([23],p.253):

$$-\frac{4\alpha^2}{15m^2}\delta(\mathbf{r}) \tag{73}$$

VI. THE OFF-MASS-SHELL VERTEX FUNCTION

Consider an electron (see Fig.2(d)) moving in a hydrogen atom, its momentum changes from p to p' via the scattering by the proton and an exchange of virtual photon with momentum k. The FDI at one-loop level reads

$$\Lambda_\mu(p',p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2} \gamma_\nu \frac{i}{\not{p}' - \not{k} - \mu} \gamma_\mu \frac{i}{\not{p} - \not{k} - \mu} \gamma^\nu \tag{74}$$

However, different from [8b] and many other literatures, not only the reduced mass μ (instead of m) of electron is used, but also a new approach will be adopted. We assume that the electron is moving off-mass-shell in the sense of (as in section IV):

$$p^2 = p'^2 = \mu^2(1-\zeta) \tag{75}$$

We still have

$$p' - p = q, \quad p \cdot q = -\frac{1}{2}q^2 \tag{76}$$

Introducing Feynman parameters $u = x + y$ and $v = x - y$, we perform a shift in the momentum integration:$k \to K = k - (p + q/2)u - (q/2)v$, thus

$$\Lambda_\mu = -ie^2[I_3\gamma_\mu + I_4] \tag{77}$$

$$I_3 = \int_0^1 du \int_{-u}^{u} dv \int \frac{d^4K}{(2\pi)^4} \frac{K^2}{(K^2 - M^2)^3} \tag{78}$$

$$M^2 = [\mu^2(1-\zeta) - \frac{q^2}{4}]u^2 + \frac{q^2}{4}v^2 + \zeta\mu^2 u \tag{79}$$

$$I_4 = \int_0^1 du \int_{-u}^{u} dv \int \frac{d^4K}{(2\pi)^4} \frac{A_\mu}{(K^2 - M^2)^3} \tag{80}$$

$$A_\mu = (4 - 4u - 2u^2)\mu^2(1-\zeta)\gamma_\mu + 2i(u^2 - u)\mu q^\nu \sigma_{\mu\nu} - (2 - 2u + \frac{u^2}{2} - \frac{v^2}{2})q^2\gamma_\mu - (2+2u)v\mu q_\mu \tag{81}$$

Set $K^2 = K^2 - M^2 + M^2$, then $I_3 = I_3' - \frac{i}{32\pi^2}$ and I_3' is only logarithmically divergent and so can be treated as in previous sections, yielding:

$$I_3' = \frac{-i}{(4\pi)^2} \int_0^1 du \int_{-u}^{u} dv \ln \frac{M^2}{\mu_1^2} \tag{82}$$

with μ_1 an arbitrary constant.

However, unlike Ref.[8b] where the calculation was conducted on the mass-shell, now the off-mass-shell integration in Eq.(82) can be performed in the approximation that $\frac{Q^2}{4\mu^2} \ll 1$ and $\zeta \ll 1$ ($Q^2 = -q^2$, Q is the three-dimensional momentum transfer) which will be enough to calculate the Lamb shift (LS). Denoting

$$a = [\mu^2(1-\zeta) + \frac{Q^2}{4}]u^2 + \zeta\mu^2 u, \quad b = \frac{Q^2}{4} \tag{83}$$

we will perform the integration with respect to v and u rigorously:

$$\int_{-u}^{u} dv \ln(a - bv^2) = 2u[\ln\mu^2 + \ln u + \ln[(1-\zeta)u + \zeta] - 4u + 2\sqrt{\frac{4a}{Q^2}} \ln\frac{1+\sqrt{Q^2/4a}u}{1-\sqrt{Q^2/4a}u} \tag{84}$$

Expanding the last term and keeping only up to the order of ζ and $Q^2/4\mu^2$, we obtain

$$\int_0^1 du \int_{-u}^{u} dv \ln(a - bv^2) \simeq \ln\mu^2 - 1 + \zeta + \frac{Q^2}{6\mu^2}(1-\zeta) \tag{85}$$

To our great pleasure, throughout the evaluation of I_4, there is no any infrared divergence which would appear in previous literatures when integrating with respect to u with lower limit zero. To avoid the infrared divergence, *e.g.*, in [8b], a cutoff was introduced at the lower limit. Now the infrared divergence disappears due to the existence of off-mass-shell parameter ζ. For example, we encounter the following integral, in which no cutoff is needed ($\lambda = (1-\zeta) + Q^2/4\mu^2 \sim 1$):

$$\int_0^1 \frac{du}{u + \zeta/\lambda} = \frac{\zeta}{\lambda} - \ln\frac{\zeta}{\lambda} \tag{86}$$

Hence, after elementary but tedious calculation, we find:

$$\begin{aligned}\Lambda_\mu(p', p) = \tfrac{\alpha}{4\pi}[\tfrac{11}{2} - \ln\tfrac{\mu^2}{\mu_1^2} - 3\zeta + 4(1+\zeta)\ln\zeta]\gamma_\mu + \tfrac{\alpha}{4\pi}\tfrac{Q^2}{\mu^2}\gamma_\mu(\tfrac{1}{6} + \tfrac{1}{2}\zeta + \tfrac{4}{3}\ln\zeta + 2\zeta\ln\zeta) \\ + i\tfrac{\alpha}{4\pi}\tfrac{q^\nu}{\mu}\sigma_{\mu\nu}(1 + 3\zeta + 2\zeta\ln\zeta)\end{aligned} \tag{87}$$

VII. CALCULATION OF LAMB SHIFT AS AN OFF-MASS-SHELL EFFECT AT ONE-LOOP LEVEL

There are three parts in Eq.(87). The first part in combination with the vertex γ_μ at tree level provides a renormalization factor as

$$Z_1^{-1} = 1 + \frac{\alpha}{4\pi}[\frac{11}{2} - \ln\frac{\mu^2}{\mu_1^2} - 3\zeta + 4(1+\zeta)\ln\zeta] \tag{88}$$

Further combination with Z_2 in Eq.(55) and Z_3 in Eq.(72) leads to a renormalized charge (at one-loop level, see Fig.2):

$$e_R = \frac{Z_2}{Z_1} Z_3^{1/2} e \tag{89}$$

However the Ward identity implies that [21–23]

$$Z_1 = Z_2 \tag{90}$$

Therefore

$$\alpha_R = \frac{e_R^2}{4\pi} = Z_3\alpha \tag{91}$$

Note that Ward identity holds not only for an electron on the mass-shell, but also for off-mass-shell case. Hence for every bound state in hydrogenlike atom with a definite value of ζ (Z_1 and Z_2 are functions of ζ), the arbitrary constant μ_1 in Eq.(88) plays a flexible role to guarantee the validity of Eq.(90) (other two constants μ_2 and μ_3 had been fixed in Eq.(58) and (72) respectively). For further discussion, see section IX.

The second part of Eq.(87) contains $Q^2\gamma_\mu$. Just like the Uehling potential in Eq.(72) (with $q^2 = -Q^2$), it contributes an effective potential of δ function type as

$$\frac{\alpha^2}{\mu^2}[-\frac{1}{6} - \frac{1}{2}\zeta - \frac{4}{3}\ln\zeta - 2\zeta\ln\zeta]\delta(\mathbf{r}) \tag{92}$$

Finally, the third part of Eq.(87) amounts to a modification of electron magnetic moment in the atom, the gyromagnetic ratio of electron reads:

$$g = 2[1 + \frac{\alpha}{2\pi}(1 + 3\zeta + 2\zeta\ln\zeta)] \tag{93}$$

We will call the anomalous part of magnetic moment $a = \frac{\tilde{\alpha}}{2\pi}$, $\tilde{\alpha} = \alpha(1+3\zeta+2\zeta\ln\zeta)$. The radiative correction on the magnetic moment of an electron has two consequences. One is a modification to the L-S coupling in a hydrogenlike atom (with charge number Z) [21, 22]:

$$H_{LS}^{rad} = 2(\frac{\tilde{\alpha}}{2\pi})\frac{\alpha Z}{4\mu^2 r^3}\sigma\cdot L \tag{94}$$

Here the electron mass has been modified from m (see, *e.g.*, [15]) to μ which can be derived from the reduced Dirac equation.

Another consequence of anomalous magnetic moment of electron exhibits itself as an additional potential of δ function type like Eq.(73)[21, 22]

$$\frac{Z\alpha\tilde{\alpha}}{2\mu^2}\delta(\mathbf{r}) \tag{95}$$

Note that Eqs.(94) and (95) are only effective to states with $L \neq 0$ and S state with $L = 0$ respectively.

Adding the results of Eqs.(94), (95) and the sum of Eqs.(73) and (92) multiplied by Z together to get all radiative corrections (at one-loop level) on electron in the hydrogenlike atom, then we get the effective potential as

$$\begin{aligned} V_{eff}^{rad} &= \tfrac{Z\alpha^2}{\mu^2}[-\tfrac{4}{3}\ln\zeta - \tfrac{1}{2}\zeta - 2\zeta\ln\zeta - \tfrac{1}{6} - \tfrac{4}{15}\tfrac{\mu^2}{m^2} + \tfrac{1}{2}(1+3\zeta+2\zeta\ln\zeta)]\delta(\mathbf{r}) \\ &+\tfrac{Z\alpha^2}{4\pi\mu^2 r^3}(1+3\zeta+2\zeta\ln\zeta)\sigma\cdot L \\ &\simeq \tfrac{Z\alpha^2}{\mu^2}[-\tfrac{4}{3}\ln\zeta + \tfrac{1}{15} + \zeta - \zeta\ln\zeta]\delta(\mathbf{r}) + \tfrac{Z\alpha^2}{4\pi\mu^2 r^3}(1+3\zeta+2\zeta\ln\zeta)\sigma\cdot L \end{aligned} \tag{96}$$

where we take $\mu^2/m^2 \approx 1$ in the Uehling potential to make the formula simpler for a semi-quantitative calculation. Eq.(96) leads to the energy modification of a bound state (with quantum numbers n, l, j) in a hydrogenlike atom:

$$\delta(\mathbf{r}) \longrightarrow |\psi_{ns}(0)|^2 = \frac{Z^3\alpha^3}{\pi n^3}\mu^3, \quad (l=0) \tag{97}$$

$$\Delta E^{rad} = \Delta E^{rad}(ns) + \Delta E_{LS}^{rad} \tag{98}$$

$$\Delta E^{rad}(ns) = \frac{Z^4\alpha^3}{\pi n^3}R_y[\frac{8}{3}\ln\frac{1}{\zeta} + \frac{2}{15} + 2\zeta(1-\ln\zeta)]\delta_{l0} \tag{99}$$

$$\Delta E_{LS}^{rad} = \frac{Z^4\alpha^3}{\pi n^3}R_y\frac{1+\zeta(3+2\ln\zeta)}{l(2l+1)(l+1)}\begin{cases} l, & (j=l+1/2) \\ -\ (l+1), & (j=l-1/2) \end{cases} \tag{100}$$

where

$$R_y = \frac{1}{2}\alpha^2\mu = \frac{\mu}{m}R_\infty \tag{101}$$

VIII. ENERGY-LEVEL DIFFERENCE IN HYDROGENLIKE ATOM: THEORY VS. EXPERIMENT

We will study some energy-level differences near the ground state of hydrogenlike atoms, where precise experimental data are available. Theoretically, the energy level is fixed primarily by the formula derived from the reduced Dirac equation (RDE), *i.e.*, Eq.(1) with m_e substituted by μ_A where the subscript A refers to atom H, D or He^+, *et al.*.:

$$\begin{aligned} E_A^{RDE} &= \mu_A[f(n,j)-1] = \frac{m_e}{1+b_A}[f(n,j)-1] \\ &= \frac{1}{1+b_A}(1.2355897\times 10^{20})[-\frac{(Z\alpha)^2}{2n^2}-\frac{(Z\alpha)^4}{3n^3}(\frac{1}{j+1/2}-\frac{3}{4n})-\cdots] \quad Hz \end{aligned} \tag{102}$$

Further recoil corrections Eq.(6) derived by previous authors will be divided into two terms:

$$\Delta E_A^{recoil-1}(n,j) = -\frac{\mu_A^2}{2M_A}[f(n,j)-1]^2 = -\frac{m_e b_A}{2(1+b_A)^3}[f(n,j)-1]^2 \tag{103}$$

$$\Delta E_A^{recoil-2}(n,j,l) = \frac{(Z\alpha)^4\mu_A^3}{2n^3 m_N^{(A)^2}}(\frac{1}{j+\frac{1}{2}}-\frac{1}{l+\frac{1}{2}})(1-\delta_{l0}) \tag{104}$$

Next comes the radiative correction calculated by QED at one-loop level, Eq.(98):

$$\Delta E_A^{rad}(n,j,l) = \frac{1}{1+b_A}\frac{Z^4}{n^3}(\frac{\alpha^3}{\pi}R_\infty)[(-\frac{8}{3}\ln\zeta+\frac{2}{15}+2\zeta(1-\ln\zeta))\delta_{l0}+\frac{1+\zeta(3+2\ln\zeta)}{2l+1}C_{jl}(1-\delta_{l0})] \tag{105}$$

where

$$C_{jl} = \begin{cases} \frac{1}{l+1}, & j=l+\frac{1}{2} \\ -\frac{1}{l}, & j=l-\frac{1}{2} \end{cases} \tag{106}$$

Finally, the finite nucleus size (NS) with radius $r_N^{(A)}$ brings a correction [10]:

$$\begin{aligned} \Delta E_A^{NS}(n,j) &= \tfrac{4}{3}(\tfrac{\mu_A}{m_e})^3\tfrac{Z^4}{n^3}(\tfrac{r_N^{(A)}}{a_\infty})^2 R_\infty\delta_{l0} \\ &= (\tfrac{1}{1+b_A})^3\tfrac{Z^4}{n^3}(4.386454987\times 10^7)[\tfrac{r_N^{(A)}(fm)}{5.2917725}]^2\delta_{l0} \quad Hz \end{aligned} \tag{107}$$

As explained in Eq.(61) with Eq.(62), the value of off-mass-shell parameter ζ in Eq.(105) can be calculated from the electron self-energy at one-loop level:

$$\frac{Z^2\alpha}{n^2} = \frac{1}{2\pi}\frac{(\zeta^{<S>} - 2\zeta^{<S>}\ln\zeta^{<S>})}{1+\alpha/3\pi} \tag{108}$$

where the superscript $< S >$ refers to "self-energy". However, we may derive the value of ζ in an alternative way. Divide the square average of four-dimensional momentum p into two parts:

$$< p^2 >=< E^2 > - < \mathbf{p}^2 > \tag{109}$$

where

$$< E^2 >= E^2 = (\mu - B)^2 \simeq \mu^2 - 2\mu B, \tag{110}$$

since the binding energy

$$B = \frac{Z^2\alpha^2}{2n^2}\mu \ll \mu \tag{111}$$

The square average of three-dimensional momentum $\mathbf{p}$, $< \mathbf{p}^2 >$, can be evaluated by the Virial theorem (*e.g.*, [15]). In a Coulomb field, an electron has potential energy $V = -\frac{Ze^2}{4\pi r}$ and kinetic energy $T = \frac{1}{2\mu}\mathbf{p}^2$. Then

$$\begin{aligned} < \mathbf{p}^2 > &= 2\mu < T >= 2\mu[-B - < V >] = 2\mu B \\ < p^2 > &= \mu^2 - 4\mu B = \mu^2(1 - \frac{4B}{\mu}) \end{aligned} \tag{112}$$

Comparing Eq.(112) with $< p^2 >= \mu^2(1 - \zeta^{<V>})$, we find

$$\zeta^{<V>} = \frac{4B}{\mu} = \frac{2Z^2\alpha^2}{n^2} \tag{113}$$

where the superscript $< V >$ refers to "Virial theorem". Table 1 gives the values of $\zeta^{<S>}$ and $\zeta^{<V>}$ with their logarithm values as well as two kinds of "average", $\zeta^{<S+V>} = \frac{1}{2}(\zeta^{<S>} + \zeta^{<V>})$ and $\zeta^{<SV>} = \sqrt{\zeta^{<S>}\zeta^{<V>}}$, to be used in Eq.(105).

Table 1. Off-mass-shell parameter ζ and $\ln\zeta$

$\frac{Z^2}{n^2}$	$\zeta^{<S>}\times 10^4$	-ln $\zeta^{<S>}$	$\zeta^{<V>}\times 10^6$	-ln $\zeta^{<V>}$	$\zeta^{<S+V>}\times 10^5$	-ln $\zeta^{<S+V>}$	$\zeta^{<SV>}\times 10^5$	$-\ln\zeta^{<SV>}$
$\frac{1}{16}$	1.546093458	8.77461	$\frac{\alpha^2}{8}=6.6564192$	11.91992886	8.0632	9.425609	3.2080284	10.34727
$\frac{1}{4}$	7.446539697	7.20259	$\frac{\alpha^2}{2}=26.6256771$	10.5336345	38.5639	7.860609	14.0808	8.86816225
1	37.73719345	5.57969	$2\alpha^2=106.502$	9.147340142	194.011	6.2450103	63.39626	7.36351521

Now we are in a position to discuss a number of cases:

(a) The so-called classic Lamb shift of hydrogen atom was measured experimentally as [10]:

$$L_H^{exp}(2S-2P)\equiv E_H(2S_{1/2})-E_H(2P_{1/2})=1057.845\ \ MHz \tag{114}$$

Theoretically, in this case ($b_H=5.446170255\times 10^{-4},\ r_N^H=r_p=0.862fm$), Eqs.(102) and (103) make no contributions while Eqs.(104) and (107) only contribute

$$\Delta E_H^{recoil-2}(2S_{1/2}-2P_{1/2})=-E_H^{recoil-2}(2,1/2,1)=-2.16156\ \ kHz \tag{115}$$

and

$$\Delta E_H^{NS}(2S-2P)=0.14525347\ \ MHz \tag{116}$$

respectively. The dominant contribution comes from Eq.(105). If using $\zeta^{<S>}$, we obtain

$$\Delta E_H^{Rad<S>}(2S-2P)=\frac{1}{1+b_H}\frac{1}{8}(4.06931316\times 10^8)[-\frac{8}{3}\ln\zeta^{<S>}+\frac{7}{15}+3\zeta^{<S>}-\frac{4}{3}\zeta^{<S>}\ln\zeta^{<S>}]$$
$$=1000.6567\ MHz \tag{117}$$

If we use another three values of $\ln\zeta$ in Table 1, we get

$$\Delta E_H^{Rad<V>}(2S-2P)\ =\ 1451.7912\ \ MHz \tag{118}$$

$$\Delta E_H^{Rad<S+V>}(2S-2P)\ =\ 1089.6513\ \ MHz \tag{119}$$

$$\Delta E_H^{Rad<SV>}(2S-2P)\ =\ 1226.0871\ \ MHz \tag{120}$$

It seems that Eq.(117) is smaller whereas Eq.(118) too large. So as an empirical rule in our semiquantitative calculation, we may use Eq.(119) to get

$$L_H^{theor.}(2S-2P) = 1089.651 + 0.145 - 0.002 = 1089.794 \quad MHz \tag{121}$$

which is larger than Eq.(114) by 3%.

(b) The Lamb shift of He^+ atom has been measured as (quoted from [27]):

$$L_{He^+}^{exp}(2S-2P) = 14041.13(17) \quad MHz \tag{122}$$

Similar to the case of hydrogen atom but with $Z=2$ and $b_{He^+} = \frac{m_e}{m_\alpha} = 0.0001371$, we find

$$\begin{aligned}
\Delta E_{He^+}^{Rad<S>}(2S-2P) &= 1.252680693 \times 10^{10} \quad Hz \\
\Delta E_{He^+}^{Rad<V>}(2S-2P) &= 2.023083608 \times 10^{10} \quad Hz \\
\Delta E_{He^+}^{Rad<S+V>}(2S-2P) &= 1.369980830 \times 10^{10} \quad Hz \\
\Delta E_{He^+}^{Rad<SV>}(2S-2P) &= 1.636521214 \times 10^{10} \quad Hz
\end{aligned} \tag{123}$$

As in the case of H atom, we take the $< S+V >$ scheme and add

$$\Delta E_{He^+}^{recoil-2}(2S-2P) = -2.165 \quad kHz \tag{124}$$

$$\Delta E_{He^+}^{NS}(2S-2P) = 4.514 \quad MHz \tag{125}$$

($r_\alpha \simeq 1.2fm$), to find the theoretical value:

$$L_{He^+}^{theor.}(2S-2P) = 13704.220 \quad MHz \tag{126}$$

which is smaller than Eq.(122) by 2.41%.

(c) The following energy-level difference is related to the "hyper Lamb shift (HLS)" [10]:

$$\Delta_H^{exp} \equiv E_H(4S) - E_H(2S) - \frac{1}{4}[E_H(2S) - E_H(1S)] = 4797.338(10) \quad MHz \tag{127}$$

Theoretically, now Eq.(102) makes the main contribution:

$$\Delta E_H^{RDE}[(4S) - \frac{5}{4}(2S) + \frac{1}{4}(1S)] = 3923.95 \;\; MHz \tag{128}$$

(The notation in parenthesis is self-evident). Eq.(103) and Eq.(105) contribute

$$\Delta E_H^{recoil-1}[(4S) - \frac{5}{4}(2S) + \frac{1}{4}(1S)] = -4.186 \;\; MHz \tag{129}$$

and

$$\begin{aligned}
\Delta E_H^{Rad<S>} &= 451.229097 \;\; MHz \\
\Delta E_H^{Rad<S+V>} &= 529.288296 \;\; MHz \\
\Delta E_H^{Rad<SV>} &= 675.907131 \;\; MHz \\
\Delta E_H^{Rad<V>} &= 903.266275 \;\; MHz
\end{aligned} \tag{130}$$

respectively. Adding a small contribution from Eq.(107)

$$\Delta E_H^{NS}[(4S) - \frac{5}{4}(2S) + \frac{1}{4}(1S)] = 0.1270967854 \;\; MHz \tag{131}$$

we get

$$\begin{aligned}
\Delta_H^{Theor.<S>} &= 4371.120197 \;\; MHz \\
\Delta_H^{Theor.<S+V>} &= 4449.179396 \;\; MHz \\
\Delta_H^{Theor.<SV>} &= 4595.798231 \;\; MHz \\
\Delta_H^{Theore.<V>} &= 3923.95 - 4.186 + 903.266275 + 0.1271 = 4823.1574 \;\; MHz
\end{aligned} \tag{132}$$

The $< V >$ scheme is only larger than Eq.(127) by 0.54%. All other schemes would be too small. So we guess that for S states $< V >$ scheme is better than $< S >$ scheme.

(d) The following energy-level difference was also measured as [10]:

$$\Delta_H'^{exp} \equiv E_H(4D_{5/2}) - E_H(2S) - \frac{1}{4}[E_H(2S) - E_H(1S)] = 6490.144(24) \;\; MHz \tag{133}$$

Theoretically, Eq.(102) also makes the main contribution:

$$\Delta E_H^{RDE}[(4D_{5/2}) - \frac{5}{4}(2S) + \frac{1}{4}(1S)] = 5747.92 \;\; MHz \tag{134}$$

while

$$\Delta E_H^{recoil-1}[(4D_{5/2}) - \frac{5}{4}(2S) + \frac{1}{4}(1S)] = -4.18611 \quad MHz \tag{135}$$

$$\Delta E_H^{recoil-2}[(4D_{5/2})] = \alpha^4 m_e (5.446170255 \times 10^{-4})^2 (\frac{1}{3} - \frac{2}{5}) = -6.9283 \quad kHz \tag{136}$$

are all small, we will have

$$\begin{aligned}
\Delta' E_H^{rad<S>}[(4D_{5/2}) - \frac{5}{4}(2S) + \frac{1}{4}(1S)] &= 302.088631 \quad MHz \\
\Delta' E_H^{rad<V>}[(4D_{5/2}) - \frac{5}{4}(2S) + \frac{1}{4}(1S)] &= 700.843464 \quad MHz \\
\Delta' E_H^{rad<S+V>}[(4D_{5/2}) - \frac{5}{4}(2S) + \frac{1}{4}(1S)] &= 369.124660 \quad MHz \\
\Delta' E_H^{rad<SV>}[(4D_{5/2}) - \frac{5}{4}(2S) + \frac{1}{4}(1S)] &= 500.131264 \quad MHz
\end{aligned} \tag{137}$$

Finally, the nucleus size effect gives

$$\Delta' E_H^{NS}[(4D_{5/2}) - \frac{5}{4}(2S) + \frac{1}{4}(1S)] = 11.62027752 \times 10^5 (\frac{1}{4} - \frac{5}{4} \times \frac{1}{8}) = 0.10894 \quad MHz \tag{138}$$

In sum, we have

$$\begin{aligned}
\Delta'^{<S>}_H &= \Delta' E_H^{RDE} + \Delta' E_H^{recoil-1} + \Delta' E_H^{recoil-2} + \Delta' E_H^{rad<S>} + \Delta' E_H^{rad<NS>} = 6045.925 \quad MHz \\
\Delta'^{<V>}_H &= 6444.679 \quad MHz \\
\Delta'^{<S+V>}_H &= 6112.961 \quad MHz \\
\Delta'^{<SV>}_H &= 6243.967 \quad MHz
\end{aligned} \tag{139}$$

which are smaller than the experimental value (133) by 6.8%, 0.7%, 5.8% and 3.8% respectively. .

(e) Experimentally, the combination of Eq.(127) with Eq.(133) yields:

$$\Delta''^{exp}_H \equiv E(4D_{5/2}) - E(4S_{1/2}) = 1692.806 \quad MHz \tag{140}$$

Then, theoretically, we have

$$\Delta''^{RDE}_{H}(4D_{5/2}-4S) \;=\; 1.823886903\times 10^{9}\;\; Hz \tag{141}$$

$$\Delta''^{recoil-1}_{H}(4D_{5/2}-4S) \;=\; 1.1008\;\; Hz \tag{142}$$

$$\Delta''^{recoil-2}_{H}(4D_{5/2}-4S) \;=\; -6.9283\;\; kHz \tag{143}$$

$$\Delta''^{NS}_{H}(4D_{5/2}-4S) \;=\; -0.0181605862\;\; MHz \tag{144}$$

and

$$\Delta''^{rad<S>}_{H}(4D_{5/2}-4S) \;=\; -149.1404661\;\; MHz \tag{145}$$

$$\Delta''^{rad<V>}_{H}(4D_{5/2}-4S) \;=\; -202.4228107\;\; MHz \tag{146}$$

$$\Delta''^{rad<S+V>}_{H}(4D_{5/2}-4S) \;=\; -160.1636366\;\; MHz \tag{147}$$

$$\Delta''^{rad<SV>}_{H}(4D_{5/2}-4S) \;=\; -175.7758676\;\; MHz \tag{148}$$

Altogether, we have

$$\begin{aligned}
\Delta''^{theore.<S>}_{H}(4D_{5/2}-4S) \;&=\; 1674.721349\;\; MHz \\
\Delta''^{theore.<S+V>}_{H}(4D_{5/2}-4S) \;&=\; 1663.716339\;\; MHz \\
\Delta''^{theore.<SV>}_{H}(4D_{5/2}-4S) \;&=\; 1648.104108\;\; MHz \\
\Delta''^{theore.<V>}_{H}(4D_{5/2}-4S) \;&=\; 1621.439105\;\; MHz
\end{aligned} \tag{149}$$

which are smaller than Eq.(140) by 1.1%, 1.7%, 2.6% and 4.2% respectively.

(f) It's time to go back to the precision data of $2S-1S$ transition in hydrogen atom as discussed in section II. Rewrite Eq.(7) as (see also [43]):

$$\Delta E^{exp}_{H}(2S-1S) = 2.46606141318734\times 10^{15}\;\; Hz \tag{150}$$

Theoretically, we have [see Eq.(17)]:

$$\Delta E^{RDE}_{H}(2S-1S) = 2.466067984\times 10^{15}\;\; Hz \tag{151}$$

$$\Delta E^{recoil-1}_{H}(2S-1S) = 22.32598676\;\; MHz \tag{152}$$

$$\begin{aligned}
\Delta E_H^{rad<S>}(2S-1S) &= -5142.081146 \;\; MHz \\
\Delta E_H^{rad<S+V>}(2S-1S) &= -5765.958928 \;\; MHz \\
\Delta E_H^{rad<SV>}(2S-1S) &= -6835.535314 \;\; MHz \\
\Delta E_H^{rad<V>}(2S-1S) &= -8541.095068 \;\; MHz
\end{aligned} \tag{153}$$

$$\Delta E_H^{NS}(2S-1S) = 11.62027752 \times 10^5 (\frac{1}{8} - 1) = -1.016774283 \;\; MHz \tag{154}$$

If taking the value of $\Delta E_H^{rad}(2S-1S)$, we get

$$\begin{aligned}
\Delta E_H^{theore.<S>}(2S-1S) &= 2.466062836 \times 10^{15} \;\; Hz \\
\Delta E_H^{theore.<S+V>}(2S-1S) &= 2.466062239 \times 10^{15} \;\; Hz \\
\Delta E_H^{theore.<SV>}(2S-1S) &= 2.466061169 \times 10^{15} \;\; Hz \\
\Delta E_H^{theore.<V>}(2S-1S) &= 2.466059464 \times 10^{15} \;\; Hz
\end{aligned} \tag{155}$$

They are larger than Eq.(150) by 1450 MHz, 826 MHz and smaller than Eq.(150) by 244 MHz, 1949 MHz respectively. Or, their discrepancies are $+5.9 \times 10^{-7}$, $+3.3 \times 10^{-7}$, -1.0×10^{-7}, -7.9×10^{-7}, respectively. This discrepancy is basically stemming from the uncertainty in the calculation of $\Delta E_H^{rad}(2S-1S)$.

(g) Let us turn to the isotope-shift of $2S-1S$ transition. Rewrite Eq.(8) as

$$\Delta E_{D-H}^{exp}(2S-1S) = 6.70994337 \times 10^{11} \;\; Hz \tag{156}$$

Theoretically, rewrite Eqs.(19) and (20) as

$$\Delta E_{D-H}^{RDE}(2S-1S) = 6.7101527879 \times 10^{11} \;\; Hz \tag{157}$$

and

$$\Delta E_{D-H}^{recoil-1}(2S-1S) = -11.176 \;\; MHz \tag{158}$$

$$\begin{aligned}
\Delta E_{D-H}^{rad<S>}(2S-1S) &= -1.399158 \quad MHz \\
\Delta E_{D-H}^{rad<V>}(2S-1S) &= -2.324028 \quad MHz \\
\Delta E_{D-H}^{rad<S+V>}(2S-1S) &= -1.568915 \quad MHz \\
\Delta E_{D-H}^{rad<SV>}(2S-1S) &= -1.859945 \quad MHz
\end{aligned} \tag{159}$$

$$\Delta E_{D-H}^{NS}(2S-1S) = -5.11384949 \quad MHz \tag{160}$$

Altogether, we find [using $< V >$ scheme in Eq.(159)]:

$$\Delta E_{D-H}^{theore.<V>}(2S-1S) = 6.709966701 \times 10^{11} \quad Hz \tag{161}$$

which is larger than Eq.(156) by 2.333 MHz or only 3.5×10^{-6}. Evidently, even Eq.(157) solely deviates from Eq.(156) by 3×10^{-5} only. And as expected, the different schemes for $\Delta E_{D-H}^{rad}(2S-1S)$ have little influence on the theoretical value, because any one of Eq.(159) is much smaller than the nucleus size effect Eq.(160) ($r_N^D = r_d = 2.115 fm$).

(h) Finally, the so-called absolute Lamb-shift of $1S$ state in hydrogen atom was determined by Weitz *et al.*[10] from the measured value Eq.(127) or (133). In our notation, using Eq.(133), we will write it as follows:

$$\begin{aligned}
L_H(1S) = \; & 4\{\Delta'^{exp}_H - \Delta E_H^{RDE}[(4D_{5/2}) - \tfrac{5}{4}(2S) + \tfrac{1}{4}(1S)] - \Delta E_H^{recoil-1}[(4D_{5/2}) - \tfrac{5}{4}(2S) + \tfrac{1}{4}(1S)] \\
& -\Delta E_H^{recoil-2}(4D_{5/2}) + \tfrac{5}{4}L_H(2S) - L_H(4D_{5/2})\}
\end{aligned} \tag{162}$$

Here the Lamb shift of $2S$ state $L_H(2S)$ can be determined from the experimental value of Eq.(114) with $L_H(2P_{1/2})$ being calculated from Eq.(105):

$$L_H(2S) = L_H^{exp}(2S-2P) - \Delta E_H^{recoil-2}(2S-2P_{1/2}) + L_H(2P_{1/2}) = 1040.901 \quad MHz \tag{163}$$

And $L_H(4D_{5/2})$ can also be calculated from Eq.(105), so

$$L_H(1S) = 8188.478 \quad MHz \tag{164}$$

which is in agreement with 8172.874(60) MHz given by [10] within an accuracy $\lesssim 0.2\%$. If we use Eq.(127) to derive $L_H(1S)$, we would have to calculate $L_H(4S)$ which is much larger than $L_H(4D_{5/2})$ and its derivation from Eq.(105) seems not reliable. Similarly, the theoretical value of $L_H(1S)$ turns out to be

$$L_H^{theore.}(1S) = \Delta E_H^{rad}(1S) + \Delta E_H^{NS}(1S) \tag{165}$$

with $\Delta E_H^{NS}(1S) = 0.14525347\ MHz$. However, the value of $\Delta E_H^{rad}(1S)$ strongly depends on the scheme we used in Eq.(105), which must be narrowed in a high-loop calculation. The theoretical prediction was given in [27] as:

$$L_H^{theor.}(1S) = 8172754(14)(32)\ kHz \tag{166}$$

Further discussions can be found in Refs. [5, 44, 45].

IX. SUMMARY AND DISCUSSION

The remarkable progress of the experimental research on energy-level differences in hydrogenlike atoms has been making this field an ideal theoretical laboratory for physics:

(a) The inevitable and successful use of reduced Dirac equation (RDE) to hydrogenlike atoms, especially to the isotope-shift of $2S-1S$ transition as reflected by Eqs.(156) through (161), is by no means an accidental fortune. It implies that the argument in section III for introducing RDE, Eq.(41), is correct to a high accuracy. In particular, the basic principle of invariance under space-time inversion Eq.(42) (with original mass m) could remain valid even for a noninertial frame. This implication has a far-reaching consequence that a generalization at the above symmetry to a localized curved space-time may be served as a possible road to quantize the general theory of relativity [16].

However, there are two realizations of potential V under the space-time inversion, Eq.(35)("scalar" type) and Eq.(42)("vector" type). While Eq.(42) does dominant in an atom like H with $m_p \gg m_e$, the remaining discrepancy of 2.333 MHz between theory and experiment [Eq.(161) versus Eq.(156)] strongly hints that an important and subtle effect had been ignored. (To consider the contribution of the deuteron polarizability merely accounts for about 20 kHz [6]). We think what neglected must be a tiny excitation of antiparticle field in the nucleus due to its interaction with electron in the CMCS. So when we reduce

the degrees of freedom of two-body system from two to one, the RDE should be modified to take account of the tiny mixture of "scalar" potentials (see the page note after Eq.(42)). We don't know how to improve RDE yet. However, an experimental evidence for the above conjecture could be the following prediction: The discrepancy between present theory (with RDE) and experiment must be smaller for the isotope shift in $2S-1S$ transition of atoms ^4He and ^3He than that of atoms H and D.

Recently, by using Dirac's method, Marsch rigorously solved the hydrogen atom as a two-Dirac particle system bound by Coulomb force [34]. His solutions are composed of positive and negative pairs, corresponding respectively to hydrogen and anti-hydrogen as expected. However, surprisingly, in the hydrogen spectrum, besides the normal type-1 solution with reduced mass μ, there is another anomalous type-2 solution with energy levels: $E'_n = Mc^2 - 2\mu c^2 + \frac{1}{2}\mu c^2(\frac{\alpha}{n})^2 + \cdots$ $(n = 1, 2, \ldots)$ and "strange enough, the type-2 ground state $(n = 1)$ does not have lowest energy but the continuum $(n = \infty)$". In our opinion, based on what we learnt from the Dirac equation and RDE, these anomalous solutions imply a positron moving in the field of proton. So all discrete states with energy E'_n are actually unbound, they should be and can be ruled out in physics either by the "square integrable condition" or the "orthogonality criterion" acting on their rigorous wave functions (for one-body Dirac equation, see [35], also p.$28-31$, 50 of [36]). On the other hand, all continuum states $(n = \infty)$ with energies lower than $Mc^2 - 2\mu c^2$ correspond to scattering wave functions with negative phase shifts , showing the repulsive force between positron and proton. (see [37], section 1.5 in [36] or section 9.5 of [15]). Marsch's discovery precisely reflects two things: (a) the negative energy state of a particle just describes its antiparticle state. (b) The Coulomb potential allows a complete set of solutions comprising of two symmetric sectors,hydrogen and antihydrogen.In the hydrogen sector, the proton remains unchanged regardless of the changing process of electron into positron under the Coulomb interaction.

The above particle-antiparticle symmetry (including Eq.(42) showing the unequal treatment between electron and nucleus), together with the parity symmetry, is hidden in the Dirac's four-component theory in covariant form so they were overlooked to some extent in the past. The advantage or flexibility of two-component noncovariant form of Dirac equation or RDE (as discussed in this paper) lies in the fact that the above two symmetries become accurate and so easily to be extended (or violated) in an explicit manner. For completeness, let us stress again that for antiparticle, one should use the momentum and energy operators

being $\mathbf{p}_c = i\nabla$ and $E_c = -i\frac{\partial}{\partial t}$ versus $\mathbf{p} = -i\nabla$ and $E = i\frac{\partial}{\partial t}$ for particle as required by the space-time inversion symmetry. The historical mission of the conception to imagine the positron as a "hole" in the sea of negative energy electrons is already over. Since the CPT invariance had been further verified [39], the relation between a particle $|a\rangle$ and its antiparticle $|\bar{a}\rangle$ is well-established as: [§]

$$|\bar{a}\rangle = CPT|a\rangle \tag{167}$$

with their wave-functions (in free motion) being respectively:

$$\langle \mathbf{x}, t|a\rangle \sim \exp[\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x} - Et)] \tag{168}$$

$$\langle \mathbf{x}, t|\bar{a}\rangle \sim \exp[-\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x} - Et)] \tag{169}$$

Note that in Eqs.(168) and (169), they have the same momentum $\mathbf{p}$ and positive energy E. Either a newly defined space-time inversion ($\mathbf{x} \to -\mathbf{x}$, $t \to -t$) or a simple change of $i \to -i$ will transform Eq.(168) into Eq.(169) (or vice versa).

(b) Throughout this paper, the electron bound in an atom is just treated like a stationary "ball" with nucleus at its center and having a (Bohr) radius ($\sim 1/\alpha m_e$). However, it is in an off-mass-shell state (In some sense, our atom model is just the opposite to J. J. Thomson's atom model 100 years ago). In fact, the electron's mass is reduced suddenly from m_e to μ in the RMCS when it is captured by a nucleus at the far remote orbit with quantum number $n \longrightarrow \infty$ and further reduced to $\mu + \delta\mu \simeq \mu - \frac{Z^2\alpha^2}{2n^2}\mu$ until n decreasing to the lowest limit $n = 1$. The Lamb shift should be viewed as a further modification on the mass of an off-mass-shell electron due to radiative correction.

Notice that the parameter Q^2 in the vertex function, Eq.(87), means the square of (three-dimensional) momentum transfer when a free electron is on its mass-shell and collides with some other particle as discussed in Ref.[8b]. By contrast, now Q^2 exhibits itself as an effective potential of δ-function type exerted by the nucleus to the bound (and so off-mass-shell) electron as shown by Eq.(92). To bind an electron to a nucleus is a nonperturbative effect. Hence we can understand why the discrepancy between $\zeta^{<S>}$ (calculated by perturbative QED at one-loop order) and $\zeta^{<V>}$ (evaluated via nonperturbative Virial theorem) is so

[§] To our knowledge, the correct definition, Eq.(167), was first given by T. D. Lee and C. S. Wu at Ann. Rev. Nucl. Sci. **15**, 381(1965). See also G. J. Ni at J. Fudan Univ. (Natural Science) No.3-4, 125(1974).

large. Fortunately, they lead to discrepancies in the calculated values of Lamb shift being not so large as shown in Section VIII. When $\zeta^{<V>}$ or $\zeta^{<S+V>}$ (or $\zeta^{<SV>}$) is substituted into the Eq.(105) which is derived from perturbative ($L = 1$) theory, we should always be aware of some theoretical inconsistency in such a semi-empirical treatment. But as a whole, we believe that the concept of Lamb shift as an off-mass-shell effect in covariant QED is basically correct.

(c) For a free on-mass-shell electron, its charge square e_R^2 will increase with the increase of Q^2 as shown by Eq.(72) (with $\mu_3 = m_e,\ q^2 = -Q^2$) and was calculated in detail in [8b], coinciding with the experimental data. Note that, however, the Ward identity $Z_1 = Z_2$ had been used. An interesting question arises for a bound electron: as its e_R^2 is not a function of Q^2, will e_R^2 change with the variation of the quantum number n? To answer this question, let us put Ward identity aside for a while and write down the renormalized $\alpha_R = \frac{e_R^2}{4\pi}$ as

$$\alpha_R = \frac{Z_2^2}{Z_1^2} Z_3 \alpha \longrightarrow \frac{Z_2^2}{Z_1^2}\alpha \tag{170}$$

Let us work in the CMCS, so $Z_2 = \frac{1}{1-B}$ and B is shown in Eq.(53) but with μ replaced by $m_e = m$. Similarly, Z_1 is given by the first part of Eq.(87) with $\mu \longrightarrow m$:

$$Z_1 \simeq 1 + \frac{\alpha}{4\pi}[\frac{11}{2} - \ln\frac{m^2}{\mu_1^2} - 3\xi + 4(1+\xi)\ln\xi] \tag{171}$$

where the off-mass-shell parameter ξ in CMCS is defined by

$$p^2 = m^2(1-\xi) = m^2(1-\eta-\zeta') = m^2(1-\eta) - m^2\zeta' = \mu^2 - m^2\zeta' = \mu^2(1-\zeta) \tag{172}$$

with

$$m^2(1-\eta) = \mu^2, \quad \eta = 1 - \frac{\mu^2}{m^2}, \quad \zeta' = \frac{\mu^2}{m^2}\zeta \tag{173}$$

and ζ is exactly that in Eq.(88). If we ignore the dependence of $(1-B)$ on ζ, Eq.(170) would give ($\zeta \ll 1$):

$$\alpha_R = \alpha[1 + \frac{2\alpha}{\pi}\ln(1+\frac{\zeta}{\eta})] \tag{174}$$

after renormalization by adjusting the arbitrary constant μ_1 so that

$$\alpha_R|_{\zeta\to 0} = \alpha \tag{175}$$

which connects to the Thomson limit $\alpha_R|_{Q\to 0} = \alpha$ for a free electron continuously but not smoothly. Then for two lowest bound states with $n = 1$ and $n = 2$, we would have (in $< V >$ scheme):

$$\alpha_R|_{n=1} \simeq \alpha(1.000433832), \quad \alpha_R|_{n=2} \simeq \alpha(1.0001123) \tag{176}$$

This would modify the Bohr energy level in hydrogenlike atom A to

$$\tilde{E}_A^{Bohr}(n) = -\frac{Z^2\alpha_n^2}{2n^2}\mu_A \tag{177}$$

and make an extra contribution to the isotope-shift as

$$\Delta\tilde{E}_{D-H}^{Bohr}(2S-1S) \simeq 726 \quad MHz \tag{178}$$

which is definitely excluded by the experiment. Hence the above consideration from Eq.(170) till Eq.(178) is wrong. We learn concretely once again that the Ward identity $Z_1 = Z_2$ is valid not only for an electron on its mass-shell, but also for off-mass-shell case. Thus we use the same value of α throughout the whole calculation.

(d) In Ref.[8b], using our RRM and new renormalization group equation (RGE) for QCD derived from it and keeping all masses of 6 quarks ($m_c = 1.031\ GeV$, $m_b = 4.326\ GeV$, $m_t = 175\ GeV$, $m_s = 200\ MeV$, $m_u = 8\ MeV$, $m_d = 10\ MeV$), we calculated the strong coupling constant $\alpha_{s_i}(Q)$ for $i = u, d, s, c, b$ respectively. Their running curves (starting from the common renormalization point $\alpha_s(M_Z) = 0.118$) follow the trend of experimental data (as shown on p.158 of [39]) quite well but separate at the low Q region. Interesting enough, each of them rises to a maximum $\alpha_{s_i}^{max}$ and then suddenly drops to zero at $Q = 0$ corresponding to a threshold energy scale E_i^{th} which could be explained as the excitation energy scale for breaking the quark pair. For example, we find $E_b^{th} = 1.13\ GeV$ which is just the hadronization energy scale of Upsilon $\Upsilon(b\bar{b})$ against its dissociation into two bosons. Experimentally, $M(\Upsilon(4s)) - M(\Upsilon) = 1.12\ GeV$ and $\Upsilon(4s) \to B^+B^-$ or $B^0\bar{B}^0$. similarly, $E_c^{th} = 0.398\ GeV$ while $M(\psi(3770)) - M(\psi(3097)) = 673\ MeV$ and $\psi(3770) \to D^+D^-$ or $D^0\bar{D}^0$. it seems that $E_s^{th} \sim 90\,MeV$ and $E_{u,d}^{th} \sim 0.4\,MeV$ are not so reliable but still reasonable.

Actually, our calculation on QCD is backed by that on QED. In [8]b, using our RRM and improved RGE, keeping all contributions from 9 charged leptons and quarks we were able to calculate the running fine-structure constant $\alpha_R(Q)$ from the renormalization point $\alpha_R(Q)|_{Q=0} = \alpha = (137.036)^{-1}$ until it coincides with the experimental value of $\alpha_{exp}(M_Z) = (128.89)^{-1}$. We fitted quark's masses as mentioned above and found no further room left for extra charged elementary particles (say, of 4th generation).

(e) In 1989, we had estimated the upper and lower bounds on Higgs mass M_H by using a nonperturbative approach in QFT — the Gaussian effective potential (GEP) method, yielding [40]:

$$76\ GeV < M_H < 170\ GeV \tag{179}$$

Like many other authors, we were bothered a lot by divergences. After a deeper study on the $\lambda\phi^4$ model by using our new RRM [8a], we restudied this problem by combination GEP with RRM, yielding[24]:

$$M_H = 138\ GeV \tag{180}$$

This is not a upper or lower bound but a prediction based on the input of experimental data:

$$\begin{aligned} &M_W = 80.359\ GeV,\ M_Z = 91.1884\ GeV, \alpha^{-1} = \frac{4\pi}{g^2 \sin^2\theta_W} = 128.89, \\ &\sin^2\theta_W = \frac{g'^2}{g^2 + g'^2} = 0.2317 \end{aligned} \tag{181}$$

where θ_W is the weak mixing (Weinberg) angle. Because of getting rid of all divergences, our calculation is well under control at every step. As now the search for Higgs particle becomes so urgent experimentally but the theoretical estimation about its mass still remains uncertain[39], we think our approach with the prediction (180) deserves to be reconsidered.

(f) Moreover, our RRM can be used in $D+1$ space-time without limitation on the space dimension D. A detailed analysis of sinh(sine)-Gordon models with $D = 1, 2$ and 3 (also using GEPM) is given by Ref.[25]. Another example is again the Lamb shift but calculated by QED in noncovariant form and by using RRM similar to that in this paper, see Appendix ([26], see also the Appendix 9A in [15]). The theoretical value (A.20) seems better than (121), showing that for dealing with the Lamb shift, the noncovariant theory may be more suitable than the covariant one at least in the lowest order.

(g) Previously, the theories for Lamb shift or generally for calculating energy levels in hydrogenlike atoms are rather complicated as reviewed in refs [1, 2] and [27], some of them

have been discussed in the Appendix of this paper. For further clarity, let us try to summarize the main obstacles, or challenges in four points:

(1) Different masses of nuclei must be taken into account;

(2) Relativistic effects of the electron (not nucleus) are important;

(3) In calculating radiative corrections, the divergence becomes severer and severer with the increase of loop number;

(4) Since nuclei's properties are different from one atom to another, to treat each atom as a two-body system individually would be a daunting task, it couldn't be rigorous eventually too. This can be clearly seen from the recent work by Marsch [34].

Facing these challenges and learning from lessons and experiences of previous authors, we see that the clue point is to replace the electron mass m by reduced mass μ and work in the noninertial frame (RMCS) throughout the entire calculation. As is well known, this can be handled in nonrelativistic QM by a mathematical trick but is impossible in relativistic case. So what we need is a new understanding on the essence of special relativity — the invariance (of theory) under the (newly defined) space-time inversion in one inertial frame. Then we are able to claim the same invariance in the RMCS with μ replacing m for establishing the RDE, ignoring a small centripetal acceleration of the nucleus in $CMCS$ (see page note after Eq.(42)). The approximation in RDE is some price paid for the much bigger gain—improving the original Dirac equation (unable to treat different nuclei) and avoiding the confusion in QED calculation because of the entanglement of two frames: CMCS with RMCS (*i.e.*, the radiative corrections are entangled with the recoil effect as we can see from previous literatures). In some senses, we jump over obstacles (1) ,(2) and (4)at the least labor cost (by constructing RDE). In the meantime, we hope RDE would help to ease the difficulty in point (3). And it's a great pleasure to see that the essential correctness of our understanding has been validated by Marsch's work as well as puzzles raised in his paper [34]. Please see also Ref [41].

As to challenge (3), only after we puzzled over the "divergence" for decades, could we suddenly realize that we misread its implication as a "large number". Rather, it means the "uncertainty". Let us look at the calculation in section IV again. Previously, many authors treated the divergent integral I in Eq.(50) by different tricks of regularization , arriving at Eq.(56). Because both A and B are divergent, it was thought that the original mass (μ here) does receive some radiative corrections (via the self-energy diagrams in Fig 2(a) and

(b)) and becomes a "renormalized" mass (μ_R here). The latter should be the observed mass in experiment or physical mass (of electron). So the original mass was called as the "bare mass", which was written into the Lagrangian density as an input parameter of QFT. Then in constructing Feynman diagrams of certain perturbative calculation, one needs to further introduce some (divergent) "counter terms" for cancelling the divergence stemming from the bare mass. Based on that understanding, the renormalization factor for wavefunction, Z_2 in Eq.(55), would be a divergent quantity too (in sharp contrast to here Eq.(59) being a fixed and finite number). Previously, In Eq.(72), while the e_R on the left handed side is the observed charge which should be finite, the e on the right handed side was regarded as a "bare charge" which, together with the Z_3, was a divergent quantity. (see Fig. 7.8 in [23]. By contrast, here both Z_3 and e are finite. Actually, here e is defined as the physical charge observed at the Thomson limit in experiment).

In our opinion, the reason why we encountered so many superfluous troubles in the past is because we overlooked what Bethe said in 1947[29]. Please read his words quoted after Eq. (A.2) in the Appendix. Let us explain via our Eq.(46). The (reduced) mass (μ) already contains some contributions from self-energy diagrams like Fig. 2(a) and (b). When we evaluate the (divergent) integral, Eq.(50), trying to find the radiative corrections on the electron, the latter is bound to confuse with that already contained in the mass. In other words, the dividing line between them is blurred inevitably. The emergence of explicit divergence is essentially a warning that the effect you want to evaluate has been entangled with the mass itself, rendering both of them uncertain. Hence the aim of so-called renormalization is nothing but to redraw the dividing line between them such that the values of mass (reconfirmed by the experiment) and the new effect (*e.g.*, the mass increment when the electron is moving off-mass-shell, Eq.(61)) can be clearly separated. In short, what we have been learning in the past decade is: At the level of QM, in the Hamiltonian like Eq. (A.1), the parameters m and e can be regarded as well-defined. But they are not so at the level of QFT. Once the calculation is made beyond the tree level, *i.e.*, with loop number $L \geq 1$, the divergence occurs and the meaning of parameters becomes obscure immediately.

We need to reconfirm all parameters contained in the Lagrangian density before they can be linked with experiments. In this sense, a model of QFT is at most an "effective field theory". According to the above point of view, we believe that our RRM just provides a natural way to carry out these processes of reconfirmation [8a], getting rid of divergences

and ambiguities. Please see also Ref [42].

(h) Last, but not least, during the learning and teaching of graduate courses on QFT for decades, we have been sharing the joy and puzzle with our students all the time. We hope that the presentation of this paper could be useful as a teaching reference to render the QFT course more understandable, interesting and attractive.

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Appendix: Comparison Between Noncovariant and Covariant Theories for Lamb Shift

1. To our knowledge, the precision theory for Lamb shift was based on a combination of noncovariant (nonrelativistic or old-fashioned) QED with covariant (or relativistic) QED as discussed in Ref.[27]. As explained clearly by Sakurai in Ref.[21], in perturbative QFT of noncovariant form, all virtual particles are "on-mass-shell". Here we wish to emphasize that a rigorous reconfirmation procedure of mass parameter was often overlooked in previous literatures. The theory for hydrogenlike atom begins with a Hamiltonian:

$$H_0 = \frac{1}{2m}\mathbf{p}^2 + \frac{1}{2m_N}\mathbf{p}^2 - \frac{Z\alpha}{r} \qquad (A.1)$$

($\mathbf{p} = -i\nabla$, see Eq.(34) in [27]). As Bethe [29] first pointed out that the effect of electron's interaction with the vector potential $\mathbf{A}$ of radiation field (see [21],[15])

$$H_{int}^{(1)} = \frac{e}{mc}\mathbf{A}\cdot\mathbf{p} \qquad (A.2)$$

should properly be regarded as already included in the observed mass m_{obs} of the electron, which is denoted by m in (A1). However, once a concrete calculation is made with (A2) being taken into account, the divergence emerges immediately. What does it mean? Mathematicians teach us that there are three implications for a divergence:

(a)It is a dimensionless number; (b)It is a large number; (c)It is uncertain. While we physicists often emphasized the point (b), we didn't pay enough attention to the points (a) and (c). We often talked about a quadratically (or linearly) divergent integral without noticing that it has a dimension (say, mass dimension) and thus meaningless in mathematics unless a mass parameter (say, m) in the integral is already fixed as a mass "unit" so that the integral can be divided by m^2 (or m) to become dimensionless. Alternatively, a logarithmically divergent integral is dimensionless and thus unaffected by the choice of unit [like Eq.(50), see also Eq.(A6) below], it just implies an uncertainty waiting to be fixed. The implication of uncertainty of a divergence will never vanish even after we introduced a cutoff by hand to curb it. For example, in a pioneering paper to explain the Lamb shift, Welton ([30], see section 9.6B in Ref.[15]) encountered an integral $I = \int_{\omega_{min}}^{\omega_{max}} \frac{d\omega}{\omega}$ with ω being the (angular) frequency of virtual photon (vacuum fluctuation). He simply set $\omega_{min} \sim mZ\alpha = Z/a$, ($a$ is Bohr radius) and $\omega_{max} \sim m$ so that $I \simeq \ln(1/Z\alpha) = 4.92$ (for $Z = 1$) which leads to an estimation of Lamb shift $L_H^{theor.}(2S_{1/2} - 2P_{1/2}) \simeq 668\ MHz$. If instead of Bohr radius, the lower cutoff is provided by the electron binding energy, one should get $I \simeq \ln(Z\alpha)^{-2}$ and $L_H^{theor.} \simeq 1336\ MHz$. (see Eq.(30) in [27]). The above arbitrariness just reflects what essential in a divergent integral is not its large magnitude ($\ln(Z\alpha)^{-1}$ is merely of the order of 10) but its uncertainty. So what important in handling the integral is not to curb (or to hide) its divergence but let the divergence exhibits itself as some arbitrary constants explicitly (as shown in section IV-VI). We will show later how to do this way for noncovariant QED.

2. While Eqs.(A1) and (A2) only describe a spinless particle, the electron has spin which endows it with the relativistic nature as shown by Eqs.(41)-(45). For two-particle system, based on Bethe-Salpeter equation, an effective Dirac equation (EDE) was derived as shown by Eq.(23) in [27]. When the electromagnetic field interaction is taken into account, the Breit potential V_{Br} was derived as shown by Eq.(35) in [27]. Then the total Breit Hamiltonian reads (Eq.(36) in [27]):

$$H_{Br} = H_0 + V_{Br} \tag{A.3}$$

However, the electron mass m' (in our notation here) appeared in EDE or V_{Br} should be that in the Dirac equation, also that in the definition of reduced mass $\mu = \frac{m'm_N}{m'+m_N}$, eventually m' could be identified with the observed mass m_{obs}, which is not equal to the m in Eq.(A1). This is because besides (A2) there is an extra interaction due to electron spin with the

radiation field:

$$H_{int}^{(2)} = \frac{ge\hbar}{4\mu c}\sigma \cdot \nabla \times \mathbf{A} \qquad (A.4)$$

($g = 2 \times 1.0011596522$ is the gyromagnetic ratio of electron, see Eq.(9A.15) in [15]). The difference between m and m' will be calculated in (A16) below. It turns out to be of the order of αm and cannot be ignored at the level of QED, especially for the explanation of Lamb shift. We guess this must be one of the reasons why all calculations based on Eq.(A3) became so complicated.

3. In noncovariant theory, the leading contribution to the Lamb shift comes from the one-photon electron self-energy. The nomenclature here is different from that in the covariant theory. Roughly speaking, so-called electron self-energy often corresponds to the vertex function in covariant theory (Fig.2(d) in this paper) or to Figs.8 and 11 in Ref.[27] and its evaluations have extended over 50 years [31]. More precisely, it is identified with the radiative insertions in the electron line and the Dirac form factor contribution. Further contributions from the Pauli form factor and the vacuum polarization [27] will add to a theoretical value of classic Lamb shift being 1050.559 MHz. If taking more high-order corrections into account, the theoretical value coincides with the experimental value 1057.845 MHz rather accurately (see Table 20 in [27]). However, the above calculation looks quite complicated due to two reasons: (a) The difficulty of dealing with two masses in two coordinate systems, the electron mass m and the reduced mass μ; (b) The introduction of an auxiliary parameter σ $[m(Z\alpha) \gg \sigma \gg m(Z\alpha)^2]$ to separate the radiative photon integration region into two parts. In the low momentum region, the Bethe Logarithm [32] in noncovariant form makes the main contribution. In the high momentum region, the evaluation is resorting to some relativistic covariant form [22]. Then two expressions are matched together to get the correct result. It seems to us that the matching trick used is doubtful because both ultraviolet and infrared divergences were ambiguously handled by some cutoff which missed the main point of renormalization—to reconfirm the mass parameter in the presence of radiative corrections as shown in section IV (covariant form) or below.

4. A simple calculation for Lamb shift in noncovariant form was proposed in Ref.[26] (see also Appendix 9A of Ref.[15]). Consider the self-energy diagram of an electron with reduced mass μ and (three-dimensional) momentum $\mathbf{p}$ in the RMCS of a hydrogenlike atom. Similar to Fig.2(a), but also different in the virtual state, now a photon has energy $\omega_k = k = |\mathbf{k}|$

while the electron has momentum $\mathbf{q} = \mathbf{p} - \mathbf{k}$ and energy $\varepsilon_q = \frac{1}{2\mu}q^2$. The electron in plane-wave state $|\mathbf{p}>$ has two interactions with the radiative field at each vertex as shown by (A2) ($m \to \mu$) and (A4), acquiring an increase in energy respectively (see FIG. 3):

$$\Delta E_p^{(j)} = \sum_i \frac{|<i|H_{int}^{(j)}|\mathbf{p}>|^2}{\varepsilon_p - \varepsilon_i}, \quad (j = 1, 2) \tag{A.5}$$

Here $\varepsilon_i = \varepsilon_q + \omega_k$ is the energy of the intermediate virtual state $|i>$. Simple evaluation leads to

$$\Delta E_p^{(1)} = -\frac{\alpha p}{\pi\mu}\int_{-1}^{1} d\eta(1-\eta^2)I, \quad I = \int_0^\infty \frac{dk}{k+\xi} \tag{A.6}$$

where $\eta = \cos\theta$ with θ being the angle between $\mathbf{k}$ and $\mathbf{p}$, $\xi = 2(\mu - p\eta)$. Like Eq.(50), we take partial derivative of the divergent integral I with respect to ξ (then the integration of k) and integrate back to I again, yielding:

$$\Delta E_p^{(1)} = b_1^{(1)}p^2 + b_2^{(1)}p^4 + \cdots \tag{A.7}$$

$$b_1^{(1)} = \frac{\alpha}{\pi\mu}(\frac{4}{3}\ln 2 + \frac{4}{3}\ln\mu - \frac{4}{3}C_1) \tag{A.8}$$

$$b_2^{(1)} = \frac{\alpha}{\pi\mu^3}(-\frac{2}{15}) \tag{A.9}$$

Note that the term $b_1^{(1)}p^2$ will combine with the kinetic energy $\frac{1}{2\mu}p^2$ of a ("spinless") electron, they are indistinguishable. The appearance of an arbitrary constant C_1 precisely reflects the fact that we cannot find the reduced mass via the valuation of $\Delta E_p^{(1)}$ in perturbation theory. So we must choose $b_1^{(1)} = 0$ to reconfirm the value of μ (which is still not the final observed mass, see below). Similar evaluation on $H_{int}^{(2)}$ (of the real electron with spin 1/2) which would induce the spin flip process between states $|\mathbf{p}, \pm\frac{1}{2}>$ and $|\mathbf{q}, \pm\frac{1}{2}>$, leads to

$$\Delta E_p^{(2)} = \frac{1}{2}\sum_{i, s_z=\pm 1/2} \frac{|<i|H_{int}^{(2)}|\mathbf{p}, s_z>|^2}{\varepsilon_p - \varepsilon_i} = -\frac{\alpha g^2}{8\pi\mu}\int_{-1}^{1} d\eta J$$

$$J = \int_0^\infty \frac{k^2 dk}{k+\xi} \tag{A.10}$$

Being a quadratically divergent integral, J needs partial derivative of third order with respect to ξ, yielding:

$$\Delta E_p^{(2)} = b_0^{(2)} + b_1^{(2)}p^2 + b_2^{(2)}p^4 + \cdots \tag{A.11}$$

$$b_0^{(2)} = \frac{g^2}{4}\frac{\alpha\mu}{\pi}[4(\ln 2 + \ln\mu) - 4C_2 - \frac{2C_3}{\mu} - \frac{C_4}{\mu^2}] \tag{A.12}$$

$$b_1^{(2)} = \frac{g^2}{4}\frac{\alpha}{\pi\mu}(\frac{4}{3}\ln 2 + 2 + \frac{4}{3}\ln\mu - \frac{4}{3}C_2) \tag{A.13}$$

$$b_2^{(2)} = \frac{g^2}{4}\frac{\alpha}{\pi\mu^3}(-\frac{1}{15}) \tag{A.14}$$

Let's manage to fix three arbitrary constants C_2, C_3 and C_4. First, the term $b_1^{(2)}p^2$ should be combined with $\frac{1}{2\mu}p^2$ term. Since μ is already fixed, further modification on μ due to electron spin should be finite and fixed. So the only possible choice of C_2 is to cancel $\ln\mu$ which is ambiguous in dimension: $C_2 = \ln\mu$, yielding

$$b_1^{(2)} = \frac{\beta}{2\mu}, \quad \beta = \frac{g^2\alpha}{2\pi}(\frac{4}{3}\ln 2 + 2) \tag{A.15}$$

Then the dimensional constants C_3 and C_4 must be chosen such that $b_0^{(2)} = 0$, implying that the starting point of this theory is the nonrelativistic Hamiltonian H_0 in Eq.(A1) without rest energy term while both masses of the nucleus and the electron (with spin) are fixed by experiments. Hence now μ acquires a modification via $b_1^{(2)}p^2$ term and becomes an observable one:

$$\mu \longrightarrow \mu_{obs} = \frac{\mu}{1+\beta} \tag{A.16}$$

However, we have to consider the relativistic energy of electron shown in Eq.(30), where the term $(-\frac{1}{8\mu^3}p^4)$ goes beyond Eq.(A1). Yet the modification of μ shown as (A16) does induce a corresponding change $-\frac{1}{8}(\frac{1}{\mu_{obs}^3} - \frac{1}{\mu^3})p^4$, which should be regarded as an invisible "background" and subtracted from the p^4 term induced by radiative corrections. (The relativistic correction is brought in via the RDE as discussed in section VIII). As a whole, the combination of contributions from $H_{int}^{(1)}$ and $H_{int}^{(2)}$ leads to

$$b_1 = b_1^{(1)} + b_1^{(2)} = b_1^{(2)} \tag{A.17}$$

and a "renormalized" b_2:

$$b_2^R = b_2^{(1)} + b_2^{(2)} + \frac{1}{8\mu^3}(3\beta + 3\beta^2 + \beta^3) \simeq \frac{\alpha}{\pi\mu_{obs}^3}(1.99808) \tag{A.18}$$

Here we only keep the lowest approximation at the last step. Hence the electron self-energy-diagram contributes a radiative correction to the energy level of the stationary state $|Z,n,l>$ in a hydrogenlike atom:

$$\Delta E^{rad}(Z,n,l) = \langle Z,n,l|b_2^R p^4|Z,n,l\rangle = [\frac{8n}{2l+1} - 3]\frac{b_2^R Z^4\alpha^4}{n^4}\mu_{obs}^4 \qquad (A.19)$$

This form, together with contributions from the vacuum polarization and nuclear size effect, gives a theoretical value for classic Lamb shift:

$$L_H^{theor.}(2S_{1/2} - 2P_{1/2}) \approx 1056.52\ MHz \qquad (A.20)$$

which is smaller than the experimental value by 0.13%. Despite its approximation involved, the above method clearly shows that so-called renormalization is nothing but a reconfirmation process of mass. We must reconfirm the mass before it could be modified via radiative corrections. Either "skipping over the first step" or "combining two steps into one" is not allowed.

5. In noncovariant theory, the (three-dimensional) momentum $\mathbf{p}$ is combined with the reduced mass μ to form a kinetic energy term $\frac{1}{2\mu}\mathbf{p}^2$ on the mass shell. Once the energy is modified whereas $\mathbf{p}$ is conserved at the vertex, μ is bound to be modified. On the other hand, in covariant theory, the electron energy turns to a component of four-dimensional momentum p and the latter is conserved at the vertex. So the (reduced) mass μ cannot be modified on the mass shell ($p^2 = \mu^2$). Therefore, the renormalization as some reconfirmation has different meaning in covariant theory versus that in noncovariant theory. We guess this is why the matching procedure of these two formalisms into one theory for Lamb shift proves so difficult.

6. Every theory in physics is not only a discovery of natural law, but also an invention of human being [33]. Hence the comparison among various theories, in many cases, is not about a problem of being right or wrong. Rather, it's about a choice of simplicity, harmony (self-consistency) and beauty. Only time can tell.

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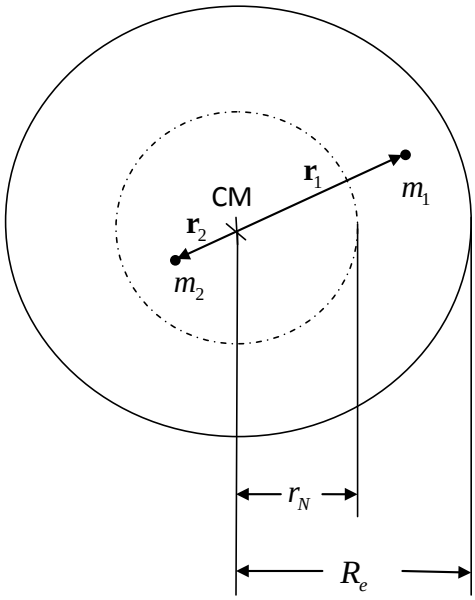

FIG. 1: A hydrogenlike atom in quantum mechanical description. The nucleus with mass m_2 occupies a small sphere with radius r_N (greatly exaggerated in the diagram) while the electron with mass m_1 spreads over a larger sphere with radius R_e (*i.e.*atomic radius). Their common center is the atom's center of mass (CM). The wavefunction $\psi(\mathbf{r})e^{-iEt}$ with $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ shows the electron's amplitude under a "fictitious measurement" [15], during which the electron and nucleus shrink into two "fictitious point particles " located at $\mathbf{r}_1$ and $\mathbf{r}_2$ simultaneously. The Coulomb potential $V(r) = -\frac{Ze^2}{r}$ between them is a static one. The probability to find the electron at $\mathbf{r}$ is $|\psi(\mathbf{r})|^2$ while that to find its momentum being $\mathbf{p}$ is $|\phi(\mathbf{p})|^2$ with $\phi(\mathbf{p})$ being the Fourier transform of $\psi(\mathbf{r})$.

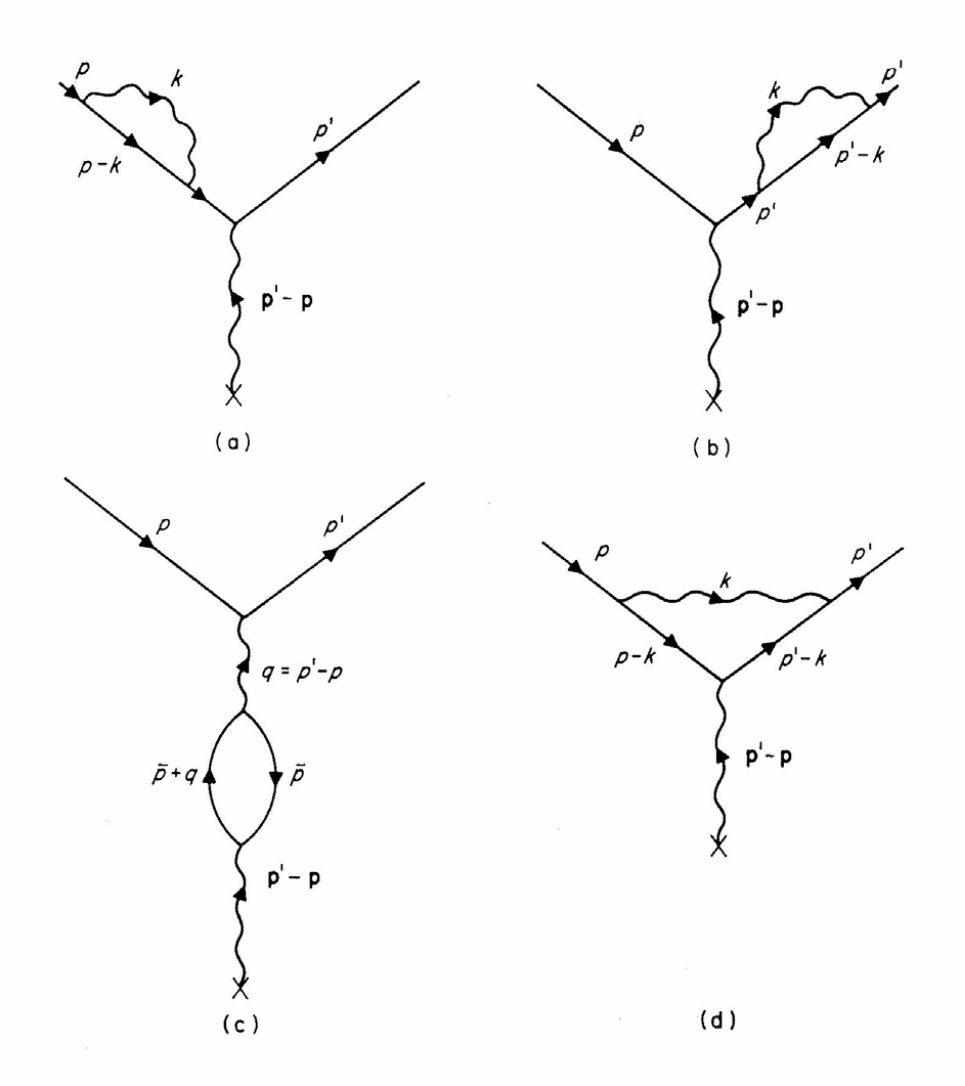

FIG. 2: Four Feynman diagrams at one-loop level (in covariant form). (a) and (b) are self-energy diagrams of the electron. (c) is vacuum polarization. (d) is vertex function. Solid lines and wavy lines refer to electron and photon respectively, while X denotes the nucleus. Here p, q and k are four-dimensional momenta.

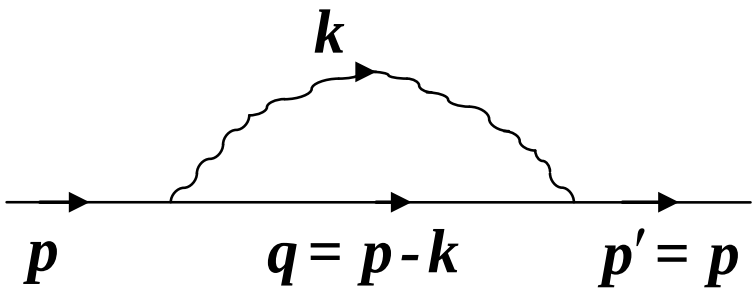

FIG. 3: The electron self-energy (radiative correction) diagram at one-loop level of perturbative QCD in noncovariant form. $H_{int}^{(1)}$ (A.2) or $H_{int}^{(2)}$ (A.4) is inserted into two vertices. Here $\mathbf{p}, \mathbf{q}$ and $\mathbf{k}$ are three dimensional momenta.